# ON THE ENUMERATION OF THE SET OF ELEMENTARY NUMERICAL SEMIGROUPS WITH FIXED MULTIPLICITY, FROBENIUS NUMBER OR GENUS

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# Notable elements

• Let $M \subseteq \mathbb{N}$ we will denote by $\langle M \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by $M$, that is,

$$\langle M \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \backslash \{0\}, a_i \in M, \lambda_i \in \mathbb{N} \text{ for all } i \in \{1, \ldots, n\}\}.$$

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• $S$ has a unique minimal system of generators $S = \langle n_1, \cdots, n_p \rangle$.

• The smallest positive integer in $S$ is called the multiplicity of $S$, denoted by $m = \mathrm{m}(S)$.

• We say that a numerical semigroup $S$ is elementary if $F(S) < 2\mathrm{m}(S)$.

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• We give algorithms that allows to compute the set of every elementary numerical semigroups with a given genus, Frobenius number and multiplicity.

• We show that sequence of cardinals of the set of elementary numerical semigroups of genus $g = 0, 1, \ldots$ is a Fibonacci sequence.

# Multiplicity and genus

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$\mathcal{E}(m, F, g)$ the set of elementary numerical semigroups with $m(S) = m$, $F(S) = F$ and $g(S) = g$.

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**Lemma**

*Let m be integer such that $m \geq 2$ and let $A \subseteq \{m+1, \ldots, 2m-1\}$. Then $\{0, m\} \cup A \cup \{2m, \rightarrow\}$ is an elementary numerical semigroup with multiplicity m. Moreover, every elementary numerical semigroup with multiplicity m is of this form.*

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## Lemma

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## Algorithm

*Input: m a positive integer.*
*Output: $\mathcal{E}(m, -, -)$.*

1) *If $m = 1$ then return $\{\mathbb{N}\}$.*
2) *If $m \geq 2$ compute the set $C = \{A \mid A \subseteq \{m+1, \ldots, 2m-1\}\}$.*
3) *Return $\{\{0, m\} \cup A \cup \{2m, \rightarrow\} \mid A \in C\}$.*

# Multiplicity and genus

## Corollary

*If m is a positive integer, then* $\#\mathcal{E}(m,-,-) = 2^{m-1}$.

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$\mathcal{E}(m,-,g) =$
$\{S \mid S$ is an elementary numerical semigroup with $\mathrm{m}(S) = m$ and $\mathrm{g}(S) = g\}$.

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## Proposition

*Let m and g be nonnegative integers with* $m \neq 0$*. Then* $\mathcal{E}(m,-,g) \neq \emptyset$ *if and only if* $\mathrm{m} - 1 \leq \mathrm{g} \leq 2(\mathrm{m}-1)$.

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$S \in \mathcal{E}(m, -, g)$ if and only if $S = \{0, m\} \cup A \cup \{2m, \rightarrow\}$ where $A \subseteq \{m+1, \ldots, 2m-1\}$ and $\#A = 2(m-1) - g$.

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## Corollary

*Let $m$ and $g$ be positive integers such that $m - 1 \leq g \leq 2\,(m-1)$. Then $\#\mathcal{E}(m, -, g) = \binom{m-1}{g-(m-1)}$.*

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We have that

$$\mathcal{E}(-,-,g) = \bigcup_{m=\lceil \frac{g}{2} \rceil + 1}^{g+1} \mathcal{E}(m,-,g).$$

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The sequence of $\#\mathcal{E}(-,-,g)$ has a Fibonacci behavior, for $g = 0, 1, 2, \ldots$

## Theorem

*If g is a positive integer, then*
$\#\mathcal{E}(-,-,g+1) = \#\mathcal{E}(-,-,g) + \#\mathcal{E}(-,-,g-1)$.

# Multiplicity and Frobenius number

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*Let m and F be positive integers. Then* $\mathcal{E}(m, F, -) \neq \emptyset$ *if and only if* $\frac{F+1}{2} \leq m \leq F + 1$ *and* $m \neq F$.

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For $F = m + i$ with $i \in \{2, \ldots, m-1\}$ and thus $S \in \mathcal{E}(m, F, -)$ if and only if there exists $A \subseteq \{m+1, \ldots, m+i-1\}$ such that
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## Corollary

*Let m and F be positive integers such that* $\frac{F+1}{2} \leq m \leq F + 1$ *and* $m \neq F$*. Then*

$$\#\mathcal{E}(m, F, -) = \begin{cases} 1 & \textit{if } m = F + 1 \\ 2^{F-m-1} & \textit{otherwise.} \end{cases}$$

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We have that

$$\mathcal{E}(-, F, -) = \bigcup_{m \in \left\{\lceil \frac{F+1}{2} \rceil, \ldots, F+1\right\} \setminus \{F\}} \mathcal{E}(m, F, -).$$

# Multiplicity and Frobenius number

**Corollary**

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Description of the behavior of sequence of cardinals of $\mathcal{E}(-, F, -)$

## Proposition

*Let F be integer greater than or equal two.*

1) *If F is odd, then* $\#\mathcal{E}(-, F+1, -) = \#\mathcal{E}(-, F, -)$.
2) *If F is even, then* $\#\mathcal{E}(-, F+1, -) = \#\mathcal{E}(-, F, -) + \#\mathcal{E}(-, F-1, -)$

# Multiplicity, Frobenius number and genus

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**Lemma**

*Let $F$ and $g$ be two positive integer. Then $g \leq F \leq 2g - 1$ if and only if $\mathcal{E}(-, F, g) \neq \emptyset$.*

# Multiplicity, Frobenius number and genus

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**Lemma**

*Let $F$ and $g$ be two positive integers such that $g \leq F \leq 2g-1$, and let $\mathcal{A}_{F,g} = \left\{ A \mid A \subseteq \left\{ \lceil \frac{F+1}{2} \rceil, \ldots, F-1 \right\} \text{ and } \#A = F - g \right\}$. Then $\mathcal{E}(-, F, g) = \left\{ \{0\} \cup A \cup \{F+1 \rightarrow\} \mid \text{ such that } A \in \mathcal{A}_{F,g} \right\}$.*

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**Corollary**

*If $F$ and $g$ are positive integers such that $g \leq F \leq 2g-1$, then $\#\mathcal{E}(-,F,g) = \binom{\lceil \frac{F}{2} \rceil - 1}{F-g}$.*

# Multiplicity, Frobenius number and genus

**Proposition**

*Let m, F and g three positive integers such that $m \geq 2$. Then $\mathcal{E}(m, F, g) \neq \emptyset$ if and only if one of the following conditions hold:*

1) $(m, F, g) = (m, m-1, m-1)$.
2) $(m, F, g) = (m, F, m)$ *and* $m < F < 2m$.
3) $m < g < F < 2m$.

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3) $m < g < F < 2m$.

For $m < g < F < 2m$ and $A \subseteq \{m+1, \ldots, F-1\}$ with $\#A = F - g - 1$ then $S = \{0, m\} \cup A \cup \{F+1, \rightarrow\} \in \mathcal{E}(m, F, g)$.

## Multiplicity, Frobenius number and genus

### Algorithm

*Input:* $m, F$ *and* $g$ *integers such that* $2 \leq m < g < F < 2m$.
*Output:* $\mathcal{E}(m, F, g)$.

1) *Compute* $C = \{A \mid A \subseteq \{m+1, \ldots, F-1\}$ *and* $\#A = F - g - 1\}$.
2) *Return* $\{\{0, m\} \cup A \cup \{F+1 \rightarrow\}$ *such that* $A \in C\}$.

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## Corollary

*Let* $m, F$ *and* $g$ *be positive integers such that* $2 \leq m < g < F \leq 2m$. *Then* $\#\mathcal{E}(m, F, g) = \binom{F-m-1}{F-g-1}$.

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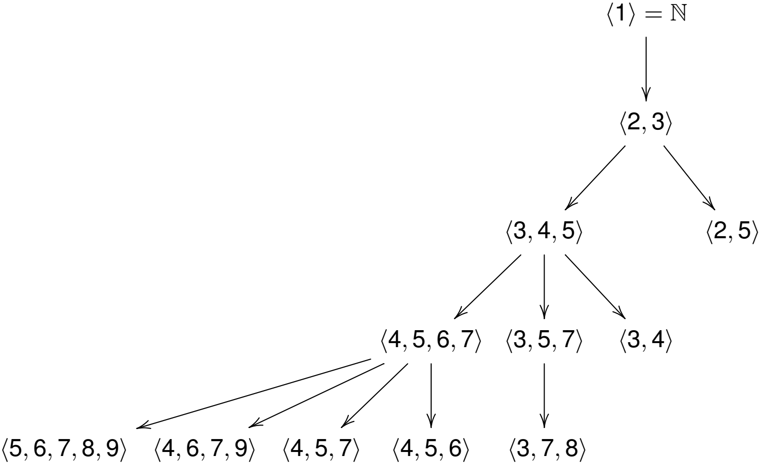

- J. C. Rosales and M.B. Branco, On the enumeration of the set of elementary numerical semigroups with fixed multiplicity, frobenius number or genus, submitted.

# Thank you.