Permutation monoids and MB-homogeneous structures

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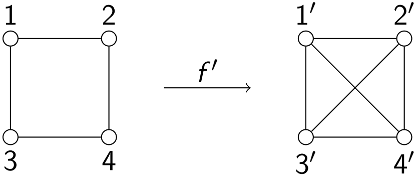
Existing work

- Ryll-Nardzewski theorem: $\aleph_0$-categorical structure $\mathcal{M} \Leftrightarrow \text{Aut}(\mathcal{M})$ is oligomorphic $\Leftrightarrow$ finitely many orbits of $\text{Aut}(\mathcal{M})$ on M^n for all $n \in \mathbb{N}$.
- Fraïssé's theorem: characterization of homogeneous structures
- Classification of countable homogeneous posets (Schmerl 1979), graphs (Lachlan and Woodrow 1980) and digraphs (Cherlin 1998)

More recently...

- Homomorphism-homogeneity, Fraïssé-style theorem for MM-homogeneous structures (Cameron and Nešetřil 2006)
- Oligomorphic transformation monoids (Mašulović and Pech 2011)
- Classification of countable homomorphism-homogeneous posets (Lockett and Truss 2014)

What is a bimorphism?



A *bimorphism* is a bijective homomorphism from a structure $\mathcal{M}$ to itself. The monoid of all such maps of $\mathcal{M}$ is written $\mathrm{Bi}(\mathcal{M})$.

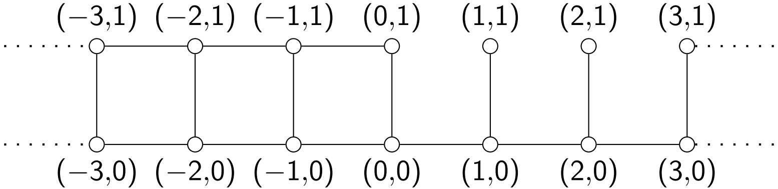
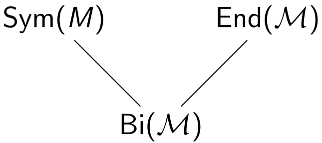


Figure: G with $\mathrm{Aut}(G) \cong 1$, $\mathrm{Bi}(G) \cong (\mathbb{N}, +)$. Bimorphisms are maps of the form $(a, x) \mapsto (a - n, x)$.

Why are we interested?



$\mathrm{Bi}(\mathcal{M})$ is an example of a *permutation monoid*. By adding the pointwise convergence topology to $\mathrm{Sym}(\mathbb{N})$, we can prove that:

Proposition

A submonoid T of $Sym(X)$ is closed under the pointwise convergence topology if and only if it is the bimorphism monoid of some structure $\mathcal{M}$.

The random graph

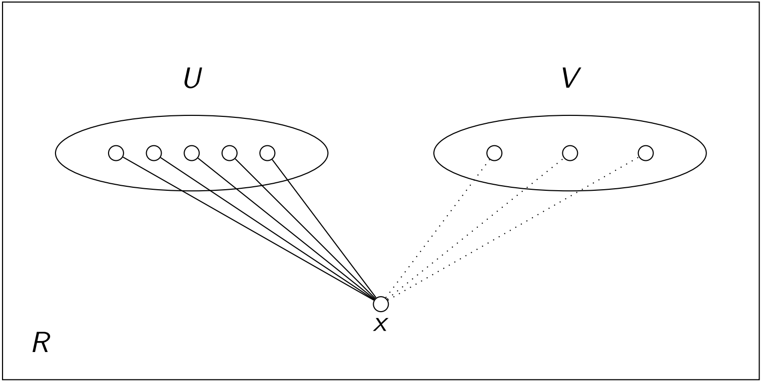


Figure: The random graph R

For R, we have that $\mathrm{Bi}(R) \neq \mathrm{Aut}(R)$, and is a closed permutation submonoid of $\mathrm{Sym}(VR)$.

Oligomorphicity of permutation monoids

Let $T \subseteq \mathrm{End}(X)$ be a transformation monoid acting on tuples X^n. Define the *strong orbit* of $\bar{x}$ to be the set

$$S(\bar{x}) = \{\bar{y} \in X^n : (\exists s, t \in T)(\bar{x}s = \bar{y} \text{ and } \bar{y}t = \bar{x})\}$$

.

Definition

We say that a permutation monoid $T \subseteq \mathrm{End}(X)$ acts *oligomorphically* on X if and only if there are finitely many strong orbits on X^n for every $n \in \mathbb{N}$. If the componentwise action of T on tuples of X is oligomorphic, we say that T is an *oligomorphic permutation monoid*.

Proposition

Let $T \subseteq End(X)$ be a permutation monoid with group of units U. If U is an oligomorphic permutation group then T is an oligomorphic permutation monoid.

Oligomorphicity of permutation monoids

Corollary

If $\mathcal{M}$ is $\aleph_0$-categorical then $Bi(\mathcal{M})$ is an oligomorphic permutation monoid.

We say that $\mathcal{M}$ is MB-homogeneous if a monomorphism between finite substructures of $\mathcal{M}$ extends to a bimorphism of $\mathcal{M}$.

Proposition

If $\mathcal{M}$ is an MB-homogeneous structure over a finite relational language, then $Bi(\mathcal{M})$ is an oligomorphic permutation monoid.

MB-homogeneity

Idea is to find analogue of Fraïssé's theorem for MB-homogeneous structures. To get a bimorphism, we need to go back and forth.

Definition

Let A, B be two relational structures. We say that an injective map $\bar{f} : A \to B$ is an *antimonomorphism* if and only if $\neg R^A(a_1, ..., a_n)$ implies $\neg R^B(a_1\bar{f}, ..., a_n\bar{f})$ for all n-ary relations R of σ.

Lemma

Let A, B be two relational structures, and suppose that $f : A \to B$ is a bijective monomorphism. Then there exists a unique antimonomorphism $\bar{f} : B \to A$ such that $f\bar{f} = 1_A$ and $\bar{f}f = 1_B$.

Extension properties

We also need to ensure MM-homogeneity (finite partial monomorphism $\mathcal{M}$ extending to a monomorphism of $\mathcal{M}$ to itself); so forward direction should mirror Cameron and Nešetřil's work.

Definition (MEP)

For all $A, B \in \mathcal{C}$ with $A \subseteq B$ and monomorphisms $f : A \to \mathcal{M}$, there exists a monomorphism $g : B \to \mathcal{M}$ extending f.

Definition (BEP)

For all $A, B \in \mathcal{C}$ with $A \subseteq B$ and antimonomorphisms $\bar{f} : A \to \mathcal{M}$, there exists a antimonomorphism $\bar{g} : B \to \mathcal{M}$ extending $\bar{f}$.

The *age* of a structure $\mathcal{M}$ is the collection of finite substructures of $\mathcal{M}$.

Proposition

Let $\mathcal{M}$ be a structure with age $\mathcal{C}$. Then $\mathcal{M}$ is MB-homogeneous if and only if $\mathcal{M}$ has the BEP and the MEP.

Amalgamation properties

Mono-amalgamation property (MAP)

For any $A, B_1, B_2 \in \mathcal{C}$, embedding $f_1 : A \to B_1$ and monomorphism $f_2 : A \to B_2$, there exists $C \in \mathcal{C}$, monomorphism $g_1 : B_1 \to C$ and embedding $g_2 : B_2 \to C$ such that $f_1 g_1 = f_2 g_2$.

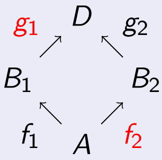


Figure: MAP

Bi-amalgamation property (BAP)

For any $A, B_1, B_2 \in \mathcal{C}$, embedding $f_1 : A \to B_1$ and antimonomorphism $\bar{f_2} : A \to B_2$, there exists $C \in \mathcal{C}$, antimonomorphism $\bar{g_1} : B_1 \to C$ and embedding $g_2 : B_2 \to C$ such that $f_1 \bar{g}_1 = \bar{f}_2 g_2$.

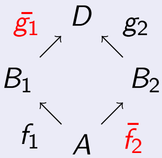


Figure: BAP

Fraïssé-style theorem for MB-homogeneous structures

Proposition

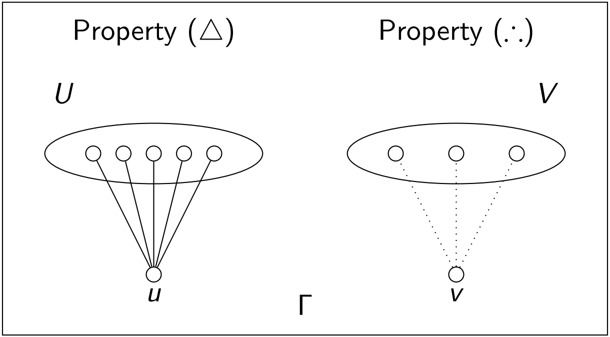
If $\mathcal{M}$ is an MB-homogeneous structure, then Age($\mathcal{M}$) has the MAP and the BAP.

Proposition

If $\mathcal{C}$ is a class of finite relational structures that is closed under isomorphisms and substructures, has countably many isomorphism types and has the JEP, MAP and BAP, then there exists an MB-homogeneous structure $\mathcal{M}$ such that Age($\mathcal{M}$).

MB-homogeneous graphs 1

We already have some examples of MB-homogeneous graphs ($R, K^{\aleph_0}, \bar{K}^{\aleph_0}, \bigsqcup_{i \in \mathbb{N}} K_i^{\aleph_0}$), however we would like to find some more; particularly those which are MB-homogeneous but not homogeneous (as the examples are).



Proposition

If Γ has both properties (△) and (∵) then Γ is MB-homogeneous.

MB-homogeneous graphs 2

Let $P = (p_n)_{n \in \mathbb{N}_0}$ be an infinite binary sequence. Define the graph $\Gamma(P)$ on the infinite vertex set $V\Gamma(P) = \{v_0, v_1, ...\}$ as follows:

- if $p_i = 0$ then $v_i \sim v_j$ for all natural numbers $j < i$;
- if $p_i = 1$ then $v_i \nsim v_j$ for all $j < i$;

where $<$ is the natural ordering on $\mathbb{N}$.

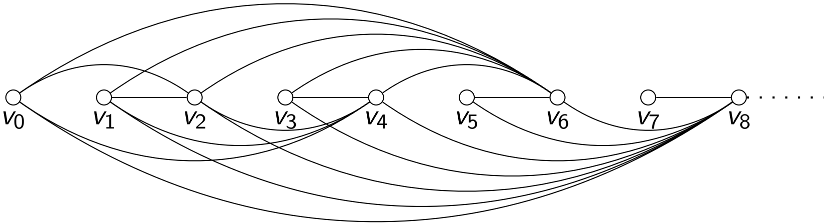


Figure: $G(A)$, with $A = (0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots)$

Bimorphism equivalence

Definition

Let Γ, Δ be two graphs. We say that Γ is *bimorphism equivalent* to Δ if there exist bijective homomorphisms $\alpha : \Gamma \to \Delta$ and $\beta : \Delta \to \Gamma$.

Proposition

Let Γ, Δ be bimorphism equivalent graphs via bijective homomorphisms $\alpha : \Gamma \to \Delta$ and $\beta : \Delta \to \Gamma$. Then Γ has properties $(\triangle)$ and $(\because)$ if and only if Δ does.

Corollary

Suppose that Γ is a graph. Then Γ has properties $(\triangle)$ and $(\because)$ if and only if it is bimorphism equivalent to R.

Uncountably many?

With so many binary sequences at our disposal, we have control over sizes of independent sets. By adding in mutually non-embeddable graphs (cycles) into the age, we can ensure that no two have the same age and are hence not isomorphic.

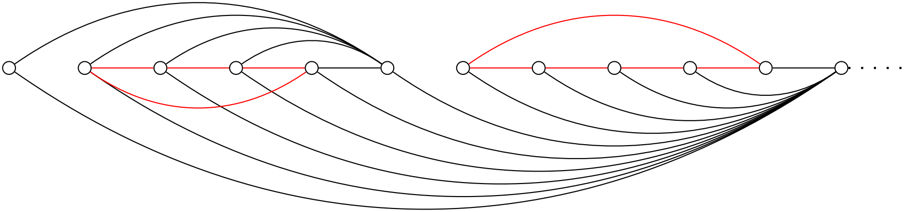


Figure: $\Gamma(PA)'$, where $P = (0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 0, ...)$ corresponds to the sequence $A = (4, 5, 6, ...)$, with added cycles highlighted in red

Uncountably many

Proposition

Suppose that $A = (a_n)_{n \in \mathbb{N}}$ *and* $B = (b_n)_{n \in \mathbb{N}}$ *are two different strictly increasing sequences of natural numbers with* $a_1, b_1 \geq 4$. *Then* $\Gamma(PA)' \ncong \Gamma(PB)'$.

Proposition

1) Any finite group H *arises as the automorphism group of an MB-homogeneous graph* Γ.
2) Any group H *that arises as the automorphism group of some countable graph* G *arises as the automorphism group of an MB-homogeneous graph.*

Reference and future work

Permutation monoids and MB-homogeneous structures, (joint with Robert Gray, in preparation).

- Find more examples of oligomorphic permutation monoids; particularly those not arising from MB-homogeneous structures- does one exist?
- Classification of MB-homogeneous graphs. Is every MB-homogeneous graph bimorphism equivalent to one of the following: R, $K^{\aleph_0}$, $\bar{K}^{\aleph_0}$ or $\bigsqcup_{i \in \mathbb{N}} K_i^{\aleph_0}$?
- Classification of countably infinite homomorphism-homogeneous graphs; finite case difficult (Rusinov and Schweitzer 2010).

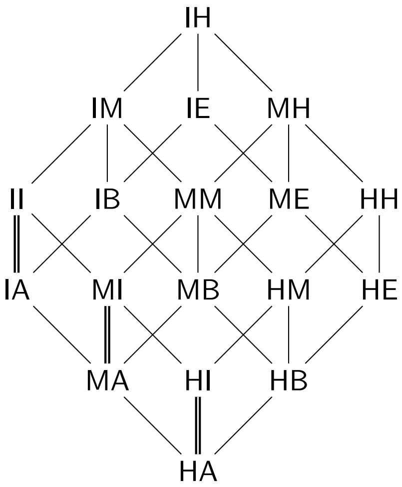


Figure: $\mathcal{H}$, the set of homomorphism-homogeneity classes partially ordered by inclusion. Lines indicate inclusion, double lines indicate equality.