Partial semigroups: Categories and Constellations

CSA 2016
In honour of Jorge Almeida and Gracinda Gomes
Lisbon, 21st June 2016

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Partial semigroups

Partial semigroup

A **partial semigroup** $(C, \cdot)$ is a non-empty set C together with a partial map

$$C \times C \to C$$

such that whenever $(xy)z$ **and** $x(yz)$ **are both defined**,

$$(xy)z = x(yz).$$

Partial semigroups: Examples

Restriction to subsets

Let S be a semigroup with $A \subseteq S$. Then $(A, \cdot)$ is a partial semigroup where

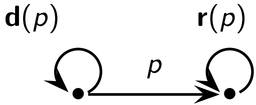
$$\begin{array}{rcl} \mathrm{dom}\, A \times A \to A & = & \{(a,b) \in A \times A : ab \in A\} \\ (a,b) & \mapsto & ab. \end{array}$$

Biordered sets:

Constellations

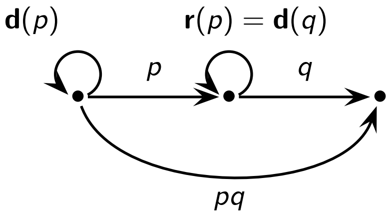
Partial semigroups
Examples related to categories

The partial semigroup $(C,\cdot)$ where $(C,\cdot,\mathbf{d},\mathbf{r})$ is a small category



Partial semigroups
Examples related to categories

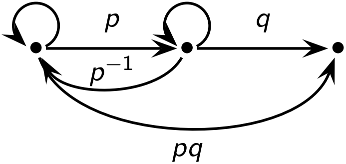
The partial semigroup $(C,\cdot)$ where $(C,\cdot,\mathbf{d},\mathbf{r})$ is a small category



Partial semigroups
Examples related to categories

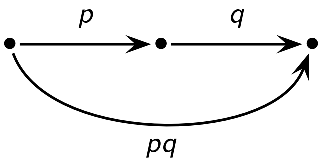
The partial semigroup $(C, \cdot)$ where $(C, \cdot, \mathbf{d}, \mathbf{r})$ is a groupoid

$\mathbf{d}(p) = pp^{-1}$ $\mathbf{r}(p) = p^{-1}p$



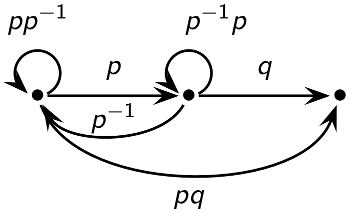
Partial semigroups
Examples related to categories

A semigroupoid $(C, \cdot)$



Partial semigroups
Examples related to categories

An inverse semigroupoid $(C, \cdot)$: for all p $\exists$ unique p^{-1} with $p = pp^{-1}p$ and $p^{-1} = p^{-1}pp^{-1}$.



Let C be a semigroupoid

Then $C^0 = C \cup \{0\}$, with undefined products equal to 0, is a semigroup.

If C is a groupoid then C^0 is primitive inverse

If C is a category then C^0 is primitive restriction

A groupoid from an inverse semigroup
The trace category

How do we pass from an inverse semigroup S to a groupoid $\mathcal{C}(S)$?

We put $\mathcal{C}(S) = (S, \cdot, \mathbf{d}, \mathbf{r})$ where

$$\mathbf{d}(a) = aa^{-1} = a^{+},\ \mathbf{r}(a) = a^{-1}a = a^{*} \text{ and } a \cdot b = ab.$$

In the 'same' way we can obtain a category from a restriction semigroup.

For inverse/restriction S we have

$$S \quad \rightarrow \quad \mathcal{C}(S) \quad \rightarrow \quad \mathcal{C}(S)^0$$

non-primitive **primitive**

How do we restore order?

D-semigroups

A unary semigroup $S = (S, \cdot, ^+)$ is a **D-semigroup** if

$$a^+a = a, (a^+)^+ = a^+ \text{ and } a^+(ab)^+ = (ab)^+ = (ab)^+a^+.$$

$E = \{a^+ : a \in S\} \subseteq E(S)$ and E is called the set of **projections**. If E is a band it is **left regular**.

The class of D-semigroups include many semigroups of interest: inverse, $\mathcal{R}$-unipotent, left ample, left restriction, left Ehresmann, C-semigroups, *glarc*, *wlqa*, left GC-lpp semigroups, etc.

Any class of D-semigroups has its left/right dual where here the operation is $a \mapsto a^*$, and two-sided versions (where we insist the sets of projections coincide)

We have the analogous classes of D-semigroupoids

Left restriction semigroups: a subvariety of D-semigroups

Let $\mathcal{PT}_X$ be equipped with the unary operation $\alpha \mapsto \alpha^+$ where α^+ is the identity map in $\operatorname{dom} \alpha$.

A unary semigroup $S = (S, \cdot, ^+)$ is left restriction if and only if S embeds into $\mathcal{PT}_X = (\mathcal{PT}_X, \cdot, ^+)$.

Consequently, left restriction semigroups are naturally partially ordered

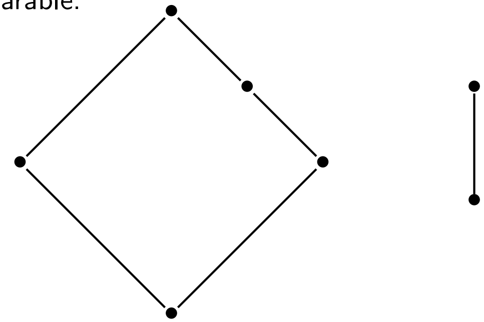
Left restriction and restriction semigroup(oid)s are defined in the canonical manner.

An inverse semigroup is restriction with $a^+ = aa^{-1}$ and $a^* = a^{-1}a$.

Note that in these cases, E is a semilattice, the **semilattice of projections**.

Local semilattices

We say that a partially ordered set is a **local semilattice** if it is disjoint union of semilattices, such that any two elements in distinct semilattices are incomparable.



Left/right/two-sided **restriction** and **inverse semigroupoids** have **local semilattices** of idempotents.

Restoring order

For inverse/restriction S we have

$$\underset{\textbf{non-primitive}}{S} \rightarrow \mathcal{C}(S) \rightarrow \underset{\textbf{primitive}}{\mathcal{C}(S)^0}$$

How do we restore order?

Ordered categories

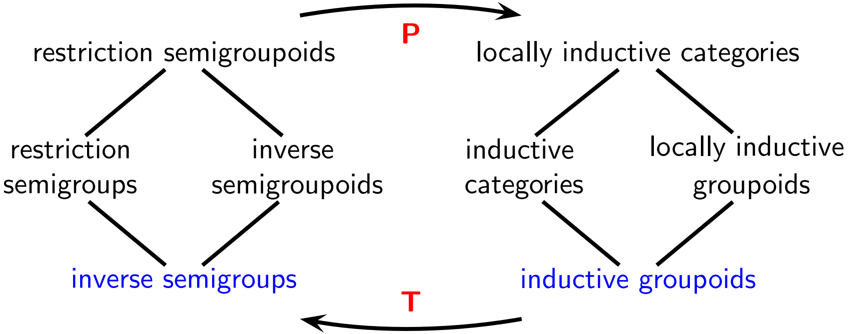
• An **ordered category** $(C, \cdot, \mathbf{d}, \mathbf{r}, \leq)$ is a category $(C, \cdot, \mathbf{d}, \mathbf{r})$ equipped with a partial order $\leq$ that is compatible with $\cdot$ and such that if $a \leq b$ then and $\mathbf{d}(a) \leq \mathbf{d}(b)$ and $\mathbf{r}(a) \leq \mathbf{r}(b)$, and possessing **restrictions** and dually **corestrictions**.

• In an **ordered groupoid**, $\leq$ must be compatible with $a \mapsto a^{-1}$.

• An **inductive groupoid (category)** is an ordered groupoid (category) in which the identities form a semilattice.

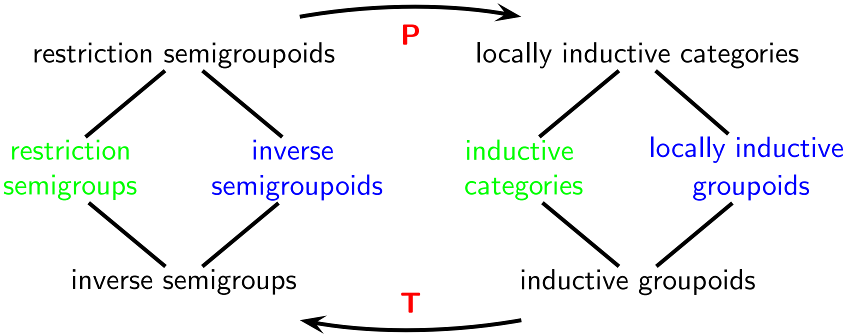
• A **locally inductive groupoid (category)** is an ordered groupoid (category) in which the identities form a local semilattice.

The Correspondence Theorems I



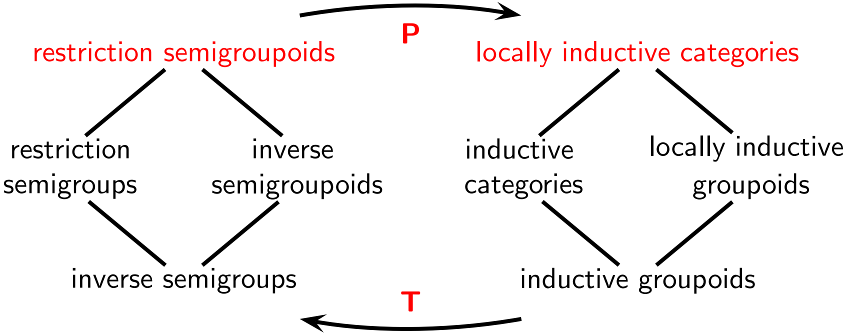
The Ehresmann-Schein-Nambooripad Theorem

The Correspondence Theorems I



M Lawson, 1991
M Lawson, 2008 (Private communication)
DeWolf and Pronk, 2015, ArXiv paper

The Correspondence Theorems I



VG: 2010

If S is restriction, then $\mathcal{C}(S) = (S, \cdot, \mathbf{d}, \mathbf{r})$ is a category **exactly as in the inverse case i.e.**

$$\mathbf{d}(a) = a^+, \mathbf{r}(a) = a^*.$$

Problem

We can't do this for a **left** restriction semigroup $S = (S, \cdot, ^+)$ as S possesses just one binary operation corresponding to 'domain'

Left restriction semigroups
McAlister theory works

Let S be a left restriction semigroup.

σ and $\widetilde{\mathcal{R}}_E$

$$\begin{array}{rcl} a\,\sigma\,b & \Leftrightarrow & ea = eb, \text{ for some } e \in E \\ a\,\widetilde{\mathcal{R}}_E\,b & \Leftrightarrow & a^+ = b^+. \end{array}$$

We say S is **proper** if

$$a\,\widetilde{\mathcal{R}}_E\,b \text{ and } a\,\sigma\,b \Leftrightarrow a = b.$$

Gomes and VG 1999; Branco, Gomes and VG 2013

Every left restriction semigroup has a proper cover. Every proper left restriction semigroup embeds into a semidirect product of a semilattice by a monoid.

Constellations

Radiating out from source/domain to...?

Definition Let P be a set, let $\cdot$ be a partial binary operation and let $^+$ be unary operation on P with image $E \subseteq E(P)$. We call $(P, \cdot, ^+)$ a **(left) constellation** if the following axioms hold:

(C1) $\exists x \cdot (y \cdot z) \Rightarrow \exists (x \cdot y) \cdot z$, in which case, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;

(C2) $\exists x \cdot (y \cdot z) \Leftrightarrow \exists x \cdot y$ and $\exists y \cdot z$;

(C3) for each $x \in P$, x^+ is the unique left identity of x in E;

(C4) $a \in P$, $y^+ \in E$, $\exists a \cdot y^+ \Rightarrow a \cdot y^+ = a$.

E is the set of **projections** of P

We construct a constellation $\mathcal{C}_X$ from $\mathcal{PT}_X$. A a restricted product is defined by

$$\alpha \cdot \beta = \begin{cases} \alpha\beta & \text{if } \operatorname{im}\alpha \subseteq \operatorname{dom}\beta \text{ i.e. if } \alpha\beta^+ = \alpha \\ \text{undefined} & \text{otherwise} \end{cases}$$

it is easy to see that $\mathcal{C}_X = (\mathcal{PT}_X, \cdot, {}^+)$ is a constellation.

Proposition: VG, Hollings, Stokes $\leq$ 2016

From any D-semigroup with $\widetilde{\mathcal{R}}_E$ a left congruence we can we can construct a constellation in a similar fashion.

Constellations
A different kind of Example

A relation $\leq$ on a set X is a **quasi-order** if it is reflexive and transitive

Example

Let T be a monoid acting by order preserving maps on a on a quasi-ordered set X. Then $X * T = (X \times T, \cdot, ^{+})$ is a constellation, where

$$\exists (x,s) \cdot (y,t) \Leftrightarrow x \leq s \cdot y \text{ and } (x,s)(y,t) = (x,st)$$

and

$$(x,t)^{+} = (x,1).$$

Let $(P, \cdot, ^+)$ be a constellation and let $\leq$ be a partial order on P.

$(P, \cdot, ^+, \leq)$ an **ordered** constellation if **natural conditions hold**

An ordered constellation is **(locally) inductive** if the projections form a (local) semilattice.

Constellations
Correspondence II

VG, Hollings: 2009

The category of left restriction semigroups is isomorphic to the category of inductive constellations.

VG, Hartmann, Lawson, Stokes: 201?

The category of left restriction semigroupoids is isomorphic to the category of locally inductive constellations.

Constellations
Relationship with categories
G and Stokes, 2016

Observation I

If $(C, \cdot, \mathbf{d}, \mathbf{r})$ is a category, then $(C, \cdot, \mathbf{d})$ is a left constellation, $(C, \cdot, \mathbf{r})$ is a right constellation, and $\mathbf{d}(P) = \mathbf{r}(P)$.

Observation II

Let P be equipped with a partial binary operation and two unary operations $\mathbf{d}, \mathbf{r}$ such that $(P, \cdot, \mathbf{d})$ is a left constellation, $(P, \cdot, \mathbf{r})$ is a right constellation and $\mathbf{d}(P) = \mathbf{r}(P)$. Then $(P, \cdot, \mathbf{d}, \mathbf{r})$ is a category.

Constellations
Relationship with categories
G and Stokes, 2016

If P is a constellation, a property an element $s \in P$ may have is that of being composable: there exists $t \in P$ such that $s \cdot t$ exists.

If P is a composable constellation, we can build a category $C(P)$ from P.

A congruence δ on a constellation P is canonical if δ separates projections and if $(a, b) \in \delta$ and $a \cdot e$ and $b \cdot e$ both exist for some $e \in E$, then a = b.

A pair (K, δ) consisting of a category K equipped with a particular canonical congruence δ upon it is a δ-category.

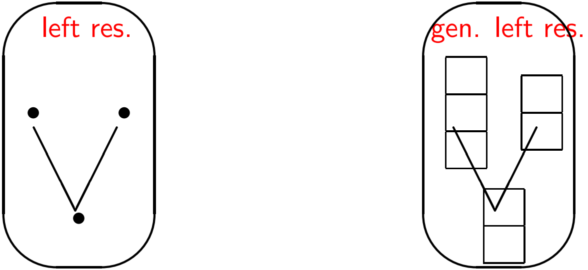
VG and Stokes, 2016

The categories of composable constellations and δ-categories are equivalent.

Generalised left restriction semigroups

These are defined as for left restriction semigroups **but** with the semilattice of projections replaced by a **left regular band**.

They form a well-behaved class of D-semigroups.



Branco, Gomes and VG, 2013

Every generalised left restriction semigroup has a proper cover that embeds in a semidirect product of a left regular band by a monoid.

D-semigroups with $\widetilde{\mathcal{R}}_E$ a left congruence: left congruence D-semigroups Correspondence IV

Stokes, 2016

The category of left congruence D-semigroups is isomorphic to the category of co-restriction constellations

$$\text{left congruence D-semigroups} \quad \underset{\longleftarrow}{\overset{\mathbf{P}}{\longrightarrow}}\!\!{}_{\mathbf{T}} \quad \text{co-restriction constellations}$$

For D-semigroups,

left restriction $\subseteq$ generalised left restriction $\subseteq$ left congruence

D-semigroups with left regular bands
Correspondence IV

Example again!

Let $S = B \rtimes T$ where B is a left regular band and T is a monoid that acts on B by morphisms. Then S is a generalised left restriction semigroup.
In $\mathbf{P}(S)$, $(e,t)^+ = (e,1)$ and

$$\exists (e,s)\cdot(f,t) \text{ if and only if } e \leq_{\mathcal{L}} s\cdot f$$

and then

$$(e,s)(f,t) = (e,st).$$

We have that $\mathbf{T}(\mathbf{P}(S)) = S$.

Questions

- A framework for constellations (products, free objects, adjoints etc)
- How to axiomatise $(E, \cdot)$ where $E = E(S)$?
- The structure of free algebras in subvarieties of D-semigroups
- When does an idempotent pure extension of a generalised left restriction semigroup T embed into a λ-semidirect product of T with a left regular band?