The semigroup of conjugacy classes of left ideals of a finite dimensional algebra

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Conjugacy classes of left ideals

Definition

Consider a conjugate action of the unit group $U(A)$ on a finite dimensional K-algebra A:

$$(g, a) \mapsto g^{-1}ag, \quad \text{for } g \in U(A), a \in A. \tag{1}$$

By **the semigroup $C(A)$ of conjugacy classes of A** we mean the set of classes $[L]$ of left ideals L in A under (1), equipped with a natural multiplication inherited from the algebra A:

$$[L_1][L_2] := [L_1 L_2].$$

Questions for this talk

1. **What is the structure of $C(A)$ and $\mathrm{K}_0[C(A)]$?**
2. **When is the semigroup $C(A)$ finite?**
3. **What information on A resides within $C(A)$?**

The structure of $C(A)$

- C(A) is periodic as the dimension of A is finite,
- $\mathcal{L}$-trivial $\Rightarrow$ Green relations $\mathcal{D}, \mathcal{J}, \mathcal{R}$ coincide,
- $\{$regular $\mathcal{J}$-classes$\} \leftrightsquigarrow \{$nonzero idempotent ideals in $A\}$,
- there exist a finite chain of ideals $0 = I_0 \subsetneq \ldots \subsetneq I_n = C(A)$ such that each factor is either nilpotent or a completely 0-simple semigroup with a unique nonzero $\mathcal{R}$-class whose elements form a right zero semigroup.
- $C(A)$ is locally finite, $K_0[C(A)]$ is basic and semiprimary.

The finiteness problem for $C(A)$

Theorem (Okniński, Renner, 2003)

If the algebra A is of **finite representation type** (finitely many isomorphism classes of finite dimensional indecomposable left modules), then $C(A)$ is finite.

Moreover, if the ground field is algebraically closed, then A is finite representation type if and only if the semigroup $C(\mathrm{M}_n(A))$ is finite for all $n \geq 1$.

The finiteness problem, when $J(A)^2 = 0$ and $\mathrm{K} = \overline{\mathrm{K}}$

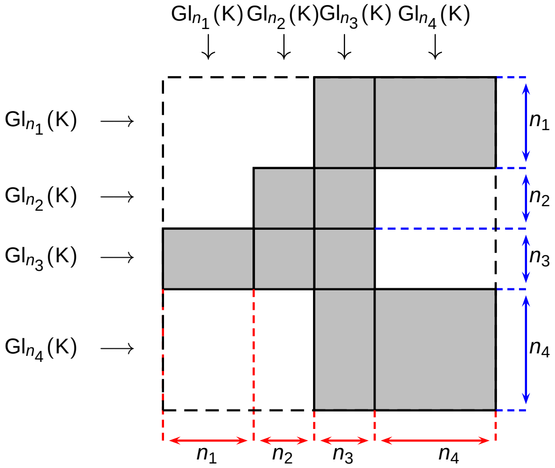
- A general fact (for any f.d. algebra): $C(A)$ is finite if and only if the number of nilpotent elements in $C(A)$ is finite.
- **Assume that** $J(A)^2 = 0, \mathrm{K} = \overline{\mathrm{K}}$ **and** $|C(A)| < \infty$
  Let $A/J(A) \simeq \prod \mathrm{M}_{n_i}(\mathrm{K})$ and let $1 = \sum e_i$, where e_i are minimal orthogonal idempotents of A that are central modulo $J(A)$. We have an isomorphism of linear spaces:

$$e_i J(A) e_j \simeq \mathrm{M}_{n_i \times n_j}(\mathrm{K}) \text{ or } 0$$

  and we can treat $J(A) = \bigoplus e_i J(A) e_j$ as a set of block matrices with some blocks equal to zero.

The finiteness problem, when $J(A)^2 = 0$ and $\mathrm{K} = \overline{\mathrm{K}}$

- Since $J(A)^2 = 0$, the conjugacy action on the radical is the action of the linear group $U(A/J(A)) \simeq \prod \mathrm{Gl}_{n_i}(\mathrm{K})$:

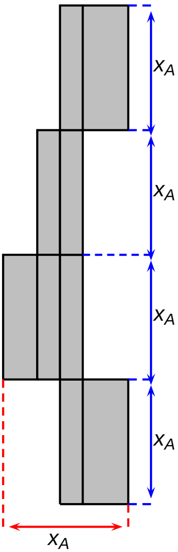


"The Matrix Problem for C(A)"

Blocks of sizes $x_A \times r_i$, where
$A/J(A) \simeq \mathrm{M}_{r_1}(\mathrm{K}) \times \ldots \times \mathrm{M}_{r_k}(\mathrm{K})$
and $x_A = r_1 + \ldots + r_k$.

The double coset action from:
the left, by $\mathrm{Gl}_{x_A}(\mathrm{K}) \times \ldots \times Gl_{x_A}(\mathrm{K})$,
the right, by $\mathrm{Gl}_{r_1}(\mathrm{K}) \times \ldots \times Gl_{r_k}(\mathrm{K})$.

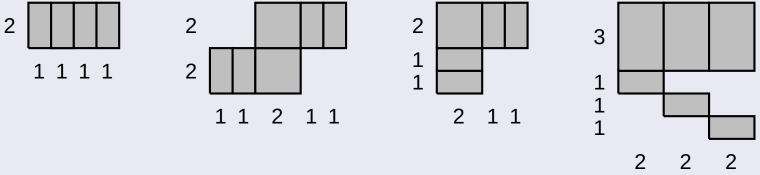
An idea: compare different skeletons by trying to „fit” one into another!



Algebras with small blocks

Theorem

If $J(A)^2 = 0$, and if $A/J(A) \simeq \prod \mathrm{M}_{n_i}(\mathrm{K})$, where $n_i \leq 2$, then $C(A)$ is finite if and only if the skeleton of **The Matrix Problem for C(A)** does not contain any of the following four skeletons.



Algebras with large blocks

Theorem

If $J(A)^2 = 0$, and if $A/J(A) \simeq \prod \mathrm{M}_{n_i}(\mathrm{K})$, where $n_i \geq 6$, then $C(A)$ is finite if and only if A is of finite representation type.

Corollary

If $J(A)^2 = 0$, then A is of finite representation type if and only if $C(\mathrm{M}_6(A))$ is finite.

$C(A)$ as an invariant of A

Question

Let A, B be finite dimensional algebras over an algebraically closed field K such that $C(A) \simeq C(B)$ as **finite semigroups**. Is A isomorphic to B? If not, then how are A and B related?

If $C(A) \simeq C(B)$, as finite semigroups, then:

- $A \simeq B$, in case when $J(A)^2 = 0$,
- $A/J(A) \simeq B/J(B)$,
- the Gabriel quivers of A and B are isomorphic (as the subquivers of the quivers of $K_0[C(A)]$ and $K_0[C(B)]$)

$C(A)$ as an invariant of A

Theorem

Let A, B be finite dimensional algebras over an algebraically closed field K. Assume that the quivers of A and B **do not have oriented cycles** and that the basic subalgebras of A and B **admit normed presentations**. If the semigroups $C(A)$ and $C(B)$ are **finite** and isomorphic, then $A \simeq B$.

Remarks:

- the algebras A, B admit multiplicative bases,
- an important example: algebras of finite representation type with acyclic quivers.
- **an open question: is any algebra of finite representation type recognizable by $C(A)$?**

Thank you for your attention!