Bilateral decompositions of some monoids of transformations

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Bilateral semidirect product

Let S and T be two semigroups and let

$$\begin{array}{rccccccc} \delta: & T & \longrightarrow & \mathcal{T}(S) & & & \\ & u & \longmapsto & \delta_u: & S & \longrightarrow & S \\ & & & & s & \longmapsto & u \centerdot s \end{array}$$

be an *anti-homomorphism of semigroups* (i.e. $(uv) \centerdot s = u \centerdot (v \centerdot s)$, for $s \in S$ and $u, v \in T$) and let

$$\begin{array}{rccccccc} \varphi: & S & \longrightarrow & \mathcal{T}(T) & & & \\ & s & \longmapsto & \varphi_s: & T & \longrightarrow & T \\ & & & & u & \longmapsto & u^s \end{array}$$

be a *homomorphism of semigroups* (i.e. $u^{sr} = (u^s)^r$, for $s, r \in S$ e $u \in T$) such that:

(SPR) $(uv)^s = u^{v \centerdot s} v^s$, *for* $s \in S$ *and* $u, v \in T$

and

(SCR) $u \centerdot (sr) = (u \centerdot s)(u^s \centerdot r)$, *for* $s, r \in S$ *and* $u \in T$.

In 1983 Kunze proved that the set $S \times T$ is a semigroup with respect to the following multiplication:

$$(s, u)(r, v) = (s(u \centerdot r), u^r v),$$

for $s, r \in S$ e $u, v \in T$. We denote this semigroup by $S \bowtie T$ and call it the *bilateral semidirect product (BSP)* of S and T (associated with δ and φ).

If S and T are monoids and the actions δ and φ *preserve the identity* (i.e. $1 \centerdot s = s$, for $s \in S$, and $u^1 = u$, for $u \in T$) and are *monoidal* (i.e. $u \centerdot 1 = 1$, for $u \in T$, and $1^s = 1$, for $s \in S$), then $S \bowtie T$ is a monoid with identity $(1, 1)$.

In 1992 Kunze proved that $\mathcal{T}(X)$, the full transformation semigroup on a finite set X, is a quocient of a BSP of $\mathcal{S}(X)$, the symmetric group on X and $\mathcal{O}(X)$, the semigroup of all order preserving full transformations on X, for some linear order on X.

1996 - Lavers gave conditions under which a BSP of two finitely presented monoids is itself finitely presented, by exhibiting explicit presentations.

Let $X_n = \{1 < 2 < \cdots < n\}$.
Denote by

- $\mathcal{T}_n$ the monoid (under composition) of all *full* transformations on X_n,
- $\mathcal{O}_n$ the submonoid of $\mathcal{T}_n$ of all *order-preserving* transformations,
- $\mathcal{O}_n^+ = \{s \in \mathcal{O}_n \mid x \leq xs, \textit{for } x \in X_n\}$ the submonoid of $\mathcal{O}_n$ of all *extensive* transformations, $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 3 & 5 & 5 \end{pmatrix} \in \mathcal{O}_5^+$,
- $\mathcal{O}_n^- = \{s \in \mathcal{O}_n \mid xs \leq x, \textit{for } x \in X_n\}$ the submonoid of $\mathcal{O}_n$ of all *co-extensive* transformations, $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 5 \end{pmatrix} \in \mathcal{O}_5^-$.

In 1992 Kunze proved that the monoid $\mathcal{O}_n$ is a quocient of a BSP of its submonoids $\mathcal{O}_n^-$ and $\mathcal{O}_n^+$:

$$\begin{array}{cccc} \mu : & \mathcal{O}_n^- \bowtie \mathcal{O}_n^+ & \twoheadrightarrow & \mathcal{O}_n \\ & (s,u) & \mapsto & su \end{array}$$

Constructing bilateral semidirect products using presentations

Let A and B be two alphabets. Suppose we have defined actions satisfying

$$b \cdot a \in A \cup \{1\}, \quad 1 \cdot a = a, \quad b \cdot 1 = 1, \quad 1 \cdot 1 = 1 \tag{1}$$

and

$$b^a \in B^*, \quad b^1 = b, \quad 1^a = 1, \quad 1^1 = 1 \ , \tag{2}$$

for $a \in A$ and $b \in B$.

First, inductively on the length of $u \in B^+$, we define

$$(ub) \cdot a = u \cdot (b \cdot a) \quad \text{and} \quad (ub)^a = u^{b \cdot a} b^a, \tag{3}$$

for $a \in A \cup \{1\}$ and $b \in B$.

Secondly, inductively on the length of $s \in A^+$, we define

$$u \cdot (as) = (u \cdot a)(u^a \cdot s) \quad \text{and} \quad u^{as} = (u^a)^s, \tag{4}$$

for $u \in B^*$ and $a \in A$.

Thus, we have well defined mappings

$$\begin{array}{rccccccc} \delta : & B^* & \longrightarrow & \mathcal{T}(A^*) & & & \\ & u & \longmapsto & \delta_u : & A^* & \longrightarrow & A^* \\ & & & & s & \longmapsto & u \centerdot s \end{array}$$

and

$$\begin{array}{rccccccc} \varphi : & A^* & \longrightarrow & \mathcal{T}(B^*) & & & \\ & s & \longmapsto & \varphi_s : & B^* & \longrightarrow & B^* \\ & & & & u & \longmapsto & u^s \ . \end{array}$$

Proposition

The mappings δ and φ are the unique left action of B^* on A^* and right action of A^* on B^*, respectively, extending the given actions.

Let δ be **any** left action of B^* on A^* and let φ be **any** right action of A^* on B^*. Let S be a monoid defined by the presentation $\langle A \mid R \rangle$ and T be a monoid defined by the presentation $\langle B \mid U \rangle$.

We say that the action δ (resp., φ) *preserves* the presentations $\langle A \mid R \rangle$ and $\langle B \mid U \rangle$ if

$$b \centerdot s = b \centerdot r \text{ in } S \quad (\text{resp., } b^s = b^r \text{ in } T),$$

for all $(s = r) \in R$ and $b \in B$, and

$$u \centerdot a = v \centerdot a \text{ in } S \quad (\text{resp. } u^a = v^a \text{ in } T),$$

for all $(u = v) \in U$ and $a \in A$.

Theorem (Fernandes,TMQ: 2011)

If a left action of B^* on A^* and a right action of A^* on B^* preserve letters and preserve the letter-irredundant presentations $\langle A \mid R \rangle$ and $\langle B \mid U \rangle$ then they induce a left action of T on S and a right action of S on T (i.e. a bilateral semidirect product $S \bowtie T$).

Theorem (Fernandes,TMQ: 2011)

Let

- M *be a monoid,*
- S *be the submonoid of* M *generated by* A*,*
- T *be the submonoid of* M *generated by* B*,*
- $S \bowtie T$ *be any bilateral semidirect product of* S *and* T *such that either the left action preserves* A **or** *the right action preserves* B*.*

If

- $A \cup B$ *generates* M *and*
- $ba = (b \cdot a)b^a$ *in* M*, for* $a \in A$ *and* $b \in B$*,*

then M *is a homomorphic image of* $S \bowtie T$*.*

Proof. We prove that the mapping

$$\begin{array}{rccc} \mu : & S \bowtie T & \longrightarrow & M \\ & (s,u) & \longmapsto & su \end{array}$$

is a surjective homomorphism.

We have an immediate consequence for semidirect products:

Corollary

let

- *M be a monoid,*
- *S be the submonoid of M generated by A,*
- *T be the submonoid of M generated by B,*
- *$S \rtimes T$ (resp., $T \ltimes S$) be any (resp., reverse) semidirect product of S and T.*

If

- *$A \cup B$ generates M and*
- *$ba = (b \centerdot a)b$ (resp., $ab = ba^b$) in M, for $a \in A$ and $b \in B$,*

then M is a homomorphic image of $S \rtimes T$ (resp., $T \ltimes S$).

Applications

For $i \in \{1, \ldots, n-1\}$, let

$$a_i = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & i+2 & \cdots & n \\ 1 & 2 & \cdots & i & i & i+2 & \cdots & n \end{pmatrix}$$

and

$$b_i = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i & i+1 & \cdots & n \\ 1 & 2 & \cdots & i-1 & i+1 & i+1 & \cdots & n \end{pmatrix}.$$

We have that:

$A = \{a_1, \ldots, a_{n-1}\}$ is a generating set of $\mathcal{O}_n^-$,
$B = \{b_1, \ldots, b_{n-1}\}$ is a generating set of $\mathcal{O}_n^+$,
so $A \cup B$ is a generating set of $\mathcal{O}_n$.

Let R^- be the set of relations

- $a_i^2 = a_i$, for $1 \leq i \leq n-1$,
- $a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} = a_{i+1} a_i$, for $1 \leq i \leq n-2$, and
- $a_i a_j = a_j a_i$, for $1 \leq i, j \leq n-1$ and $|i-j| \geq 2$,

and R^+ be the set of relations

- $b_i^2 = b_i$, for $1 \leq i \leq n-1$,
- $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} = b_i b_{i+1}$, for $1 \leq i \leq n-2$, and
- $b_i b_j = b_j b_i$, for $1 \leq i, j \leq n-1$ and $|i-j| \geq 2$.

then

$$\mathcal{O}_n^- = \langle A \mid R^- \rangle \qquad \text{and} \qquad \mathcal{O}_n^+ = \langle B \mid R^+ \rangle \ .$$

Notice that this presentations are letter-irredundant.

Consider the left action δ of B^* on A^* and the right action φ of A^* on B^* that extend the actions:

$$b_j \centerdot a_i = \begin{cases} 1 & \text{if } j = i+1 \\ a_i & \text{otherwise} \end{cases} \quad \text{e} \quad b_j^{a_i} = \begin{cases} 1 & \text{if } j = i \\ b_j & \text{otherwise} \end{cases},$$

for $1 \leq i, j \leq n-1$.

Notice that δ and φ preserve letters.

We have

- The actions δ and φ preserve the presentations $\langle A \mid R^- \rangle$ and $\langle B \mid R^+ \rangle$,
- $b_j a_i = (b_j \centerdot a_i) b_j^{a_i}$ in $\mathcal{O}_n$, for $1 \leq i, j \leq n-1$.

Theorem (Kunze:1992; Fernandes,TMQ: 2011)

The monoid $\mathcal{O}_n$ is a homomorphic image of $\mathcal{O}_n^- \bowtie \mathcal{O}_n^+$.

Similarly to $\mathsf{V} \rtimes \mathsf{W}$ and $\mathsf{V} \ltimes \mathsf{W}$ we define

$$\mathsf{V} \bowtie \mathsf{W} = \langle M \bowtie N \mid M \in \mathsf{V}, N \in \mathsf{W} \rangle.$$

Clearly, $\mathsf{V} \rtimes \mathsf{W} \subseteq \mathsf{V} \bowtie \mathsf{W}$ and $\mathsf{V} \ltimes \mathsf{W} \subseteq \mathsf{V} \bowtie \mathsf{W}$.

Corollary

$\mathsf{O} \subsetneq \mathsf{J} \bowtie \mathsf{J}$ ($\subseteq \mathsf{A}$).

It is easy to show that $\mathsf{J} \bowtie \mathsf{J} \subseteq \mathsf{A}$. By the other hand we have $\mathsf{R} \subseteq \mathsf{J} \rtimes \mathsf{R} = \mathsf{J} \rtimes \mathsf{J}$ (a particular instance of a result of Almeida and Weil) $\mathsf{J} \rtimes \mathsf{J} \subseteq \mathsf{J} \bowtie \mathsf{J}$ and Higgins showed that $\mathsf{R} \not\subseteq \mathsf{O}$.

$\mathcal{O}_n \rtimes \mathcal{C}_2 \twoheadrightarrow \mathcal{OD}_n$

$\mathcal{C}_2 \ltimes \mathcal{O}_n \twoheadrightarrow \mathcal{OD}_n$

$\rightsquigarrow$

$\mathsf{OD} \subseteq \mathsf{O} \rtimes \mathsf{Ab}_2$

$\mathsf{OD} \subseteq \mathsf{Ab}_2 \ltimes \mathsf{O}$

$\mathcal{C}_n \bowtie \mathcal{O}_n \twoheadrightarrow \mathcal{OP}_n \quad \rightsquigarrow \quad \mathsf{OP} \subseteq \mathsf{Ab} \bowtie \mathsf{O}$

$\mathcal{C}_n \bowtie \mathcal{OD}_n \twoheadrightarrow \mathcal{OR}_n \quad \rightsquigarrow \quad \mathsf{OR} \subseteq \mathsf{Ab} \bowtie \mathsf{OD}$

$\mathcal{OP}_n \rtimes \mathcal{C}_2 \twoheadrightarrow \mathcal{OR}_n$

$\mathcal{C}_2 \ltimes \mathcal{OP}_n \twoheadrightarrow \mathcal{OR}_n$

$\rightsquigarrow$

$\mathsf{OR} \subseteq \mathsf{OP} \rtimes \mathsf{Ab}_2$

$\mathsf{OR} \subseteq \mathsf{Ab}_2 \ltimes \mathsf{OP}$

$\mathcal{D}_{2n} \bowtie \mathcal{O}_n \twoheadrightarrow \mathcal{OR}_n \quad \rightsquigarrow \quad \mathsf{OR} \subseteq \mathsf{Dih} \bowtie \mathsf{O}$

Recall that $X_n = \{1 < 2 < \cdots < n\}$.
Denote by

- $\mathcal{POI}_n$ the monoid of all *parcial* injective and order-preserving transformations on X_n, $\alpha = \begin{pmatrix} 2 & 3 & 5 & 8 \\ 1 & 2 & 7 & 10 \end{pmatrix} \in \mathcal{POI}_{10}$
- $\mathcal{POI}_n^+$ the submonoid of $\mathcal{POI}_n$ of all extensive transformations, $\alpha = \begin{pmatrix} 1 & 4 & 5 & 6 & 9 \\ 1 & 5 & 6 & 8 & 10 \end{pmatrix} \in \mathcal{POI}_{10}^+$
- $\mathcal{POI}_n^-$ the submonoid of $\mathcal{POI}_n$ of all co-extensive transformations, $\alpha = \begin{pmatrix} 2 & 7 & 8 \\ 1 & 3 & 7 \end{pmatrix} \in \mathcal{POI}_{10}^-$
- $\mathcal{PODI}_n$ the monoid of all *parcial* injective and order-preserving or order-reversing transformations on X_n.

Proposition (Fernandes,TMQ: 2016)

The monoid $\mathcal{POI}_n$ is a homomorphic image of $\mathcal{POI}_n^- \bowtie \mathcal{POI}_n^+$.

Thus, with regard to pseudovarieties we have

Corollary

$\mathsf{POI} \subsetneq (\mathsf{J} \cap \mathsf{Ecom}) \bowtie (\mathsf{J} \cap \mathsf{Ecom}) \not\subseteq \mathsf{Ecom}$.

Proposition (Fernandes,TMQ: 2016)

The monoid $\mathcal{PODI}_n$ is a homomorphic image of $\mathcal{POI}_n \rtimes \mathcal{C}_2$.

Corollary

$\mathsf{PODI} \subseteq \mathsf{POI} \rtimes \mathsf{Ab}_2$.

CONJECTURE

$\mathsf{PODI} = \mathsf{POI} \rtimes \mathsf{Ab}_2$.

Denote by

- $\mathcal{I}_n$ the symmetric inverse semigroup on X_n, i.e. the monoid, under composition of maps, of all *partial* permutations of X_n,

- $\mathcal{DP}_n = \{\alpha \in \mathcal{I}_n \mid |x\alpha - y\alpha| = |x - y| \; x, y \in \mathrm{Dom}(\alpha)\}$ the submonoid of $\mathcal{I}_n$ of all *isometries*, for example, in $\mathcal{DP}_9$

$$\begin{pmatrix} 1 & 3 & 7 \\ ? & 5 & ? \end{pmatrix}: \quad \begin{pmatrix} 1 & 3 & 7 \\ 3 & 5 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 7 \\ 7 & 5 & 1 \end{pmatrix}$$

- $\mathcal{ODP}_n$ the submonoid of $\mathcal{DP}_n$ of all order-preserving isometries,

- $\mathcal{ODP}_n^+$ the submonoid of $\mathcal{ODP}_n$ of all extensive isometries,

- $\mathcal{ODP}_n^-$ the submonoid of $\mathcal{ODP}_n$ of all co-extensive isometries.

Proposition (Fernandes,TMQ: 2016)

The monoid $\mathcal{ODP}_n$ *is a homomorphic image of* $\mathcal{ODP}_n^- \bowtie \mathcal{ODP}_n^+$.

Proposition (Fernandes,TMQ: 2016)

The monoid $\mathcal{DP}_n$ *is a homomorphic image of* $\mathcal{ODP}_n \rtimes \mathcal{C}_2$.