Categories Determined by Inverse Semigroups

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Introduction

Several categories arise in the structure theory of inverse semigroups and also of regular semigroups.

For example the well known structure theorem of Schein(cf. [10]) for inverse semigroups is based on the inductive groupoid associated with the inverse semigroup. Here a groupoid is a category in which all morphisms are isomorphisms.

Several other categories associated with inverse semigroups can be seen in literature, for example J.E. Pin and Stuart Margolis [6], M.V. Lawson[3], AR Rajan[8] etc.

One category of interest in studying structure of regular semigroups is the category of principal left [right] ideals of the semigroup introduced by K.S.S. Nambooripad (cf.[5]).
These categories have been abstractly characterised as normal categories. Normal categories are the basic categories considered in the cross connection theory of regular semigroups, which is a generalization of Grillet's cross connection (cf.[1]).

One interesting question associated with study of normal categories is regarding tne characterization of normal categories arising from various classes of regular semigroups. We consider this question for certain classes of inverse semigroups in this talk.

The contents of the talk are the following.

1. Normal categories associated with regular semigroups
2. Normal categories arising from inverse semigroups
3. Normal categories from E-unitary inverse semigroups
4. Characterization of normal categories arising fron Clifford semigroups

Normal Categories from Regular Semigroups

The normal category $\mathcal{L}(S)$ of principal left ideals of a regular semigroup S is described by Nambooripad (cf.[5]) as follows.

The objects of $\mathcal{L}(S)$ are principal left ideals Sa with $a \in S$.

Since S is a regular semigroup every principal left[right] ideal is generated by an idempotent. So $v\mathcal{L}(S) = \{Se : e \in E(S)\}$ where $E(S)$ is the set of all idempotents of S.

For Se, Sf in $v\mathcal{L}(S)$, a morphism from Se to Sf is a map

$$\rho(e, v, f) : Se \to Sf$$

which maps $x \in Se$ to $xv \in Sf$ where $v \in eSf$.

It may be noted that $e\rho(e, v, f) = v$ and corresponding to each $v \in eSf$ there is a unique morphism $\rho(e, v, f) : Se \to Sf$.

For morphisms $\rho(e, u, f) : Se \to Sf$ and $\rho(f, v, h) : Sf \to Sh$ the product is defined as the composition of the given maps and this becomes $\rho(e, uv, h) : Se \to Sh$.

A normal category is abstractly described as follows.

The description begins with the concept of a category with subobjects which is a pair $(\mathcal{C}, \mathcal{P})$ where $\mathcal{C}$ is a small category and $\mathcal{P}$ is a subcategory of $\mathcal{C}$ in such a way that $v\mathcal{P} = v\mathcal{C}$ and $\mathcal{P}$ is a strict preorder.

In this case $v\mathcal{C} = v\mathcal{P}$ becomes a partially ordered set with respect to the preorder structure of $v\mathcal{P}$.
That is for $a, b \in v\mathcal{C}$

$$a \leq b \text{ if there is a morphism from } a \text{ to } b \text{ in } \mathcal{P}.$$

The morphisms of $\mathcal{P}$ are called inclusions and are denoted as

$$j(a, b) : a \to b.$$

Another concept used in the description of normal category is that of normal cone.
A normal cone in a category with subobjects $(\mathcal{C}, \mathcal{P})$ is a map

$$\gamma : v\mathcal{C} \to \mathcal{C}$$

such that

1. there is an object c such that for every $a \in v\mathcal{C}, \ \gamma(a) : a \to c$. This c is called the vertex of γ and is usually denoted by c_γ
2. if $a, b \in v\mathcal{C}$ are such that $a \leq b$ then $\gamma(a) = j(a, b)\gamma(b)$
3. There is at least one $a \in v\mathcal{C}$ such that $\gamma(a)$ is an isomorphism.

Definition 2.1

A category with sub objects $(\mathcal{C}, \mathcal{P})$ is said to be a normal category if the following hold.

(1.) *Every inclusion $j = j(a, b) : a \to b$ has a right inverse $q : b \to a$ in $\mathcal{C}$. Such a morphism q is called a retraction in $\mathcal{C}$.*

(2) *Every morphism $f \in \mathcal{C}$ has a factorization in the form*

$$f = quj$$

where q is a retraction, u is an isomorphism and j is an inclusion. Such a factorization is called a normal factorization.

(3) *For each $c \in v\mathcal{C}$ there exists a normal cone γ such that $\gamma(c) = 1_c$.*

Each normal category $\mathcal{C}$ gives rise to a regular semigroup denoted by $T\mathcal{C}$, which is the semigroup of all normal cones in $\mathcal{C}$ with a suitable composition.

Moreover the normal category $\mathcal{C}$ is isomorphic to the normal category $\mathcal{L}(S)$ where $S = T\mathcal{C}$.

Inverse Semigroups

Now we consider the special properties of the normal category $\mathcal{L}(S)$ when S is an inverse semigroup.

The following theorem describes the major properties of the normal category $\mathcal{L}(S)$ of an inverse semigroup S.

Theorem 3.1

Let S be an inverse semigroup and $\mathcal{L}(S)$ be the normal category of principal left ideals of S. Then

1. *Every morphism $\rho(e, u, f) : Se \to Sf$ in $\mathcal{L}(S)$ has a unique normal factorization.*
2. *If $Se \subseteq Sf$ then there is unique retraction from Sf to Se.*
3. *Let $j = \rho(e, e, f) : Se \to Sf$ be an inclusion in $\mathcal{L}(S)$ and $q = \rho(f, g, g) : Sf \to Sg$ be a retraction in $\mathcal{L}(S)$ such that jq is an isomorphism. Then $e = g$ and jq is identity.*
4. *For each object $Se \in \mathcal{L}(S)$ there is a normal cone γ in $\mathcal{L}(S)$ whose vertex is Se and $\gamma(Se) = 1_{Se}$.*
5. *For each $a \in S$ let ρ^a be the cone with vertex $Se = Sa$ and components $\rho^a(Sg) = \rho(g, ga, e)$. Then ρ^a is a normal cone in $\mathcal{L}(S)$ and ρ^a has exactly one isomorphism component.*

E-Unitary Inverse Semigroups

It may be noted that all normal cones in $\mathcal{L}(S)$ are not of the form ρ^a for some $a \in S$.

We now show that if S is an $E-$unitary inverse semigroup then idempotent normal cones in $\mathcal{L}(S)$ are of the form ρ^e and $\mathcal{L}(S)$ has other nice properties.

An inverse semigroup S is said to be $E-$unitary if for $e \in E(S)$ and $s \in S$ if $es \in E(S)$ or $se \in E(S)$ then $s \in E(S)$ (cf[2]).

We have the following special properties for $\mathcal{L}(S)$ in this case.

Theorem 3.2

Let S be an $E-$unitary inverse semigroup, $e \in E(S)$ and γ be a normal cone in $\mathcal{L}(S)$ with vertex Se such that $\gamma(Se) = 1_{Se}$. Then $\gamma = \rho^e$.

Another property of the normal category arising from E-unitary inverse semigroups is the following.

Theorem 3.3

Let S be an $E-$unitary inverse semigroup and γ be a normal cone in $\mathcal{L}(S)$. Then if $\gamma(a)$ and $\gamma(b)$ are isomorphisms then $a = b$.

E-Unitary Clifford Semigroups

Another class of inverse semigroups for which the associated normal category has a simple characterization is the class of E-uniary Clifford semigroups.

Clifford semigroups are inverse semigroups which are semilattices of groups.

The following theorem characterises normal categories determined by E-unitary Clifford semigroups.

Theorem 3.4

Let $\mathcal{C}$ be a normal category such that $v\mathcal{C}$ is a semilattice with respect to the the partial order given by the inclusion in $\mathcal{C}$ satisfying the following.

1. *Normal factorizations and retractions are unique.*
2. *A morphism $f : a \to b$ is an isomorphism if and only if $a = b$.*
3. *If $a \leq b$ then for every $u \in \mathcal{C}(a, b)$ the map $u \mapsto j(a, b)uq(b, a)$ is a one to one map from $\mathcal{C}(b, b)$ to $\mathcal{C}(a, a)$.*
4. *Let $j : a \to b$ be an inclusion and $q : b \to c$ be a retraction in $\mathcal{C}$ then $jq = q_1 j_1$ for some retraction q_1 and inclusion j_1.*

Then the semigroup of normal cones $S = T\mathcal{C}$ is an E-unitary Clifford semigroup and $\mathcal{C}$ is isomorphic to $\mathcal{L}(S)$. Conversely if S is any $E-$unitary Clifford semigroup then $\mathcal{L}(S)$ satisfies the conditions above.

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References III

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