Homological lemmas for Schreier extensions of monoids

Manuela Sobral

CMUC, Universidade de Coimbra

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Joint work with Nelson Martins-Ferreira and Andrea Montoli

An action of a group B on a group X is a group homomorphism $\varphi \colon B \to \operatorname{Aut}(X)$.

Group actions correspond bijectively to split extensions: given a split extension

$$X \xrightarrow{k} A \xrightarrow{s} B,$$

the corresponding action is given by

$$b \bullet x = s(b) \cdot x \cdot s(b)^{-1}.$$

Conversely, given an action, the corresponding split extension is obtained via the semidirect product construction.

Monoid actions vs Schreier split epimorphisms

Similarly, an action of a monoid B on a monoid X can be defined as a monoid homomorphism $\varphi \colon B \to \text{End}(X)$. To which split extensions do they correspond?

Definition

A split epimorphism
$$X \xrightarrow{k} A \xrightarrow{s} B$$
 of monoids is a Schreier
split epimorphism if, for any $a \in A$, there exists a unique
 $x \in Ker(f)$ such that $a = x \cdot sf(a)$.

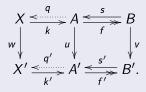
Equivalently, there exists a unique map $q: A \to X$ such that $a = q(a) \cdot sf(a)$. Given a Schreier split epimorphism, the corresponding action is defined by

$$b \bullet x = q(s(b) \cdot x).$$

The converse is given, again, by a semidirect product construction.

Theorem (Bourn, Martins-Ferreira, Montoli, S.)

Consider the following commutative diagram, where the two rows are Schreier split extensions:



We have that

 (i) u is a surjective homomorphism if and only if both v and w are;

(ii) u is a monomorphism if and only if both v and w are;
(iii) u is an isomorphism if and only if both v and w are.

Schreier reflexive relations

An internal relation on a monoid B is a submonoid of the product $B \times B$. By considering the homomorphic inclusion

$$R \rightarrowtail B \times B$$

and by composing it with the two projections of the product, we get two parallel homomorphisms

$$R \xrightarrow[p_2]{p_1} B,$$

that are the first and the second projection of the relation.

Definition

An internal reflexive relation of monoids

$$\mathsf{R} \xrightarrow[\rho_2]{\rho_1}{\stackrel{\rho_1}{\xrightarrow{}}} B$$

is a Schreier reflexive relation if the split epimorphism (R, B, p_1, ρ) is a Schreier one.

Theorem (Bourn, Martins-Ferreira, Montoli, S.)

Any Schreier reflexive relation is transitive. It is a congruence if and only if $Ker(p_1)$ is a group.

Example

The usual order between natural numbers:

$$\mathcal{O}_{\mathbb{N}} \xrightarrow[\stackrel{\rho_1}{\stackrel{}{\underbrace{\leftarrow}\rho \longrightarrow}} \mathbb{N},$$

where

$$\mathcal{O}_{\mathbb{N}} = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x \leq y\},\$$

is a Schreier order relation.

Special Schreier extensions

Definition

A homomorphism $f: A \rightarrow B$ is special Schreier if the kernel congruence

$$Eq(f) \xrightarrow[f_2]{f_1} A$$

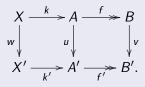
is a Schreier congruence.

This is equivalent to having a partial division on A: if $f(a_1) = f(a_2)$, then there exists a unique $x \in \text{Ker}(f)$ such that $a_2 = x \cdot a_1$. In particular, Ker(f) is a group. If $f: A \to B$ is a surjective special Schreier homomorphism, then it is the cokernel of its kernel. Hence we get an extension

$$X \xrightarrow{k} A \xrightarrow{f} B$$

Theorem (Bourn, Martins-Ferreira, Montoli, S.)

Consider the following commutative diagram, where the two rows are special Schreier extensions:



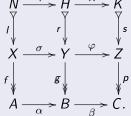
We have that

 (i) u is a surjective homomorphism if and only if both v and w are;

(ii) u is a monomorphism if and only if both v and w are;(iii) u is an isomorphism if and only if both v and w are.

Theorem

Consider the following commutative diagram, where the three columns are special Schreier extensions: $N \xrightarrow{\eta} H \xrightarrow{\lambda} K$



- If the first two rows are special Schreier extensions, then the lower also is;
- if the last two rows are special Schreier extensions, then the upper also is;
- if $\varphi \sigma = 0$ and the first and the last rows are special Schreier extensions, then the middle also is.