Algebras of Ehresmann semigroups and categories

Itamar Stein

Bar-Ilan University

June 21, 2016

Inverse semigroups

There are two equivalent definitions for inverse semigroups:

- For every $a \in S$ there is a unique $b \in S$ with $aba = a$ and $bab = b$
- S is regular and $E(S)$ is a semilattice (= commutative band)

There is a natural partial order:

$$a \leq b \iff a = eb \iff a = bf \quad e, f \in E(S)$$

Inductive groupoids

Let G be a groupoid (= category for which every morphism is an isomorphism).
For every morphism $m : a \to b$ we denote $\mathbf{d}(m) = a$ and $\mathbf{r}(m) = b$.
Let $\leq$ be some partial order on morphisms of G.
For an object a we say that $a \leq m$ ($m \leq a$) if $1_a \leq m$ ($m \leq 1_a$).

Inductive groupoids

Definition

$(G, \leq)$ is called an inductive groupoid if it satisfies some axioms including:

- For every morphism m and object e such that $e \leq \mathbf{d}(m)$, there exists a unique *restriction*, denoted $(e \mid m)$: A morphism such that

$$\mathbf{d}((e \mid m)) = e, \quad (e \mid m) \leq m$$

- For every morphism m and object e such that $e \leq \mathbf{r}(m)$, there exists a unique *co-restriction*, denoted $(m \mid e)$: A morphism such that

$$\mathbf{r}((m \mid e)) = e, \quad (m \mid e) \leq m$$

ESN Theorem

Theorem (Ehresmann-Schein-Nambooripad)

The category of all inverse semigroups is isomorphic to the category of all inductive groupoids.

$$a \in S \quad \Longrightarrow G(a) : aa^{-1} \to a^{-1}a$$

$$G(a) \cdot G(b) = \begin{cases} G(ab) & a^{-1}a = bb^{-1} \\ \text{undefined} & \text{otherwise} \end{cases}$$

E-Ehresmann semigroups

.

Definition

Let S be a semigroup and $E \subseteq S$ be a subsemilattice. S is called E-Ehresmann if the following holds:

- For every $a \in S$ there is a miniaml $e \in E$ such that $ea = a$ (denoted a^+).
- For every $a \in S$ there is a minimal $e \in E$ such that $ae = a$ (denoted a^*).
- The following identities hold: $(ab)^+ = (ab^+)^+$ and $(ab)^* = (a^*b)^*$.

E-Ehresmann semigroups form a variety of bi-unary semigroups.
There are two natural partial orders:

$$a \leq_r b \Leftrightarrow a = eb \quad e \in E$$

and

$$a \leq_l b \Leftrightarrow a = be \quad e \in E$$

Examples of E-Ehresmann semigroups

Examples

- Inverse semigroup S with $E = E(S)$ ($a^+ = aa^{-1}$ and $a^* = a^{-1}a$).
- Monoid M with $E = \{1\}$.
- $S = \mathrm{PT}_n$ or $S = \mathrm{B}_n$ with $E = \{1_A \mid A \subseteq \{1, \ldots, n\}\}$ the set of partial identities.

Ehresmann categories

Let C be a category with two partial orders on morphisms, $\leq_r$ and $\leq_l$.

Definition

$(C, \leq_r, \leq_l)$ is called an Ehresmann category if it satisfies some axioms including:

- For every morphism m and object e such that $e \leq_r \mathbf{d}(m)$, there exists a unique *restriction*, denoted $(e \mid m)$: A morphism such that

$$\mathbf{d}((e \mid m)) = e, \quad (e \mid m) \leq_r m$$

- For every morphism m and object e such that $e \leq_l \mathbf{r}(m)$, there exists a unique *co-restriction*, denoted $(m \mid e)$: A morphism such that

$$\mathbf{r}((m \mid e)) = e, \quad (m \mid e) \leq_l m$$

Ehresmann semigroups - categories Isomorphism

Theorem (Lawson, '91)

The category of all E-Ehresmann semigroups is isomorphic to the category of all Ehresmann categories.

$$a \in S \quad \Longrightarrow C(a) : a^{+} \to a^{*}$$

$$C(a) \cdot C(b) = \begin{cases} C(ab) & a^{*} = b^{+} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Algebras of semigroups and categories

Definition

Let S be a semigroup and $\mathbb{K}$ be a commutative ring. The semigroup algebra $\mathbb{K}S$ is the free module of all linear combinations

$$\{\sum_{i=1}^{n} k_i s_i \mid k_i \in \mathbb{K},\ s_i \in S\}$$

with multiplication being extension of the semigroup multiplication.

Definition

Let C be a category and $\mathbb{K}$ be a commutative ring. The category algebra $\mathbb{K}C$ is the free module of all linear combinations

$$\{\sum_{i=1}^{n} k_i m_i \mid k_i \in \mathbb{K},\ m_i \text{ morphism}\}$$

with multiplication being extension of

$$m \cdot m' = \begin{cases} mm' & \mathbf{r}(m) = \mathbf{d}(m') \\ 0 & \mathbf{r}(m) \neq \mathbf{d}(m') \end{cases}$$

Isomorphism of algebras

Theorem (Steinberg, '06)

Let S be an inverse semigroup with $E(S)$ finite and let G be the corresponding inductive groupoid. Then for any commutative ring $\mathbb{K}$ there is an isomorphism

$$\mathbb{K}S \cong \mathbb{K}G$$

Isomorphism of algebras

	Categories isomorphism	Algebras isomorphism
Inverse semigroups - Inductive groupoids	ESN	Steinberg
Ehresmann semigroups - Ehresmann categories	Lawson	

Isomorphism of algebras

	Categories isomorphism	Algebras isomorphism
Inverse semigroups - Inductive groupoids	ESN	Steinberg
Ehresmann semigroups - Ehresmann categories	Lawson	?

Isomorphism of algebras

	Categories isomorphism	Algebras isomorphism
Inverse semigroups - Inductive groupoids	ESN	Steinberg
Ehresmann semigroups - Ehresmann categories	Lawson	✓

Isomorphism of algebras

Theorem (IS)

Let S be an E-Ehresmann semigroup with E finite and let C be the corresponding Ehresmann category. Then for any commutative ring $\mathbb{K}$ there is an isomorphism

$$\mathbb{K}S \cong \mathbb{K}C$$

Actually, it is enough to require E to be principally finite (=any principal down ideal is finite).

Remark

Guo and Chen proved this result for finite ample semigroups ('12).
(inverse semigroups $\subseteq$ ample semigroups $\subseteq$ Ehresmann semigroups)

EI-categories

Definition

A category C is called an EI-category if every endomorphism is an isomorphism.

Algebras of EI-categories are better understood than general category algebras (the radical and ordinary quiver are known over "good" fields).

Ehresmann EI-categories

Lemma

- *Let S be an E-Ehresmann semigroup. The corresponding Ehresmann category is EI if and only if $a^+ = a^*$ implies $a\mathcal{H}a^+$ for every $a \in S$.*
- *Necessary condition: E is a maximal semilattice in S.*

Ehresmann EI-categories

Examples

- PT_n and $(2,1,1)$ - subalgebras of PT_n.
- $E(S)$-Ehresmann semigroups. (In particular: adequate semigroups, ample semigroups).

Maximal semisimple image (If time allows)

Let A be a $\mathbb{K}$-algebra where $\mathbb{K}$ is a field. By the Wedderburn-Malcev theorem If $\operatorname{Top} A = A/\operatorname{Rad} A$ (=the maximal semisimple image of A) is seperable then it is always isomorphic to a subalgebra of A. However, if $A = \mathbb{K}S$ is a semigroup algebra, it is usually not the case that $\operatorname{Top} A \cong \mathbb{K}R$ where R is a subsemigroup of S.

Theorem (Steinberg, '08)

Let $\mathbb{K}$ be a field with $\operatorname{char} \mathbb{K} = 0$. Let S be a finite semigroup with $E(S)$ a semilattice. Denote by $R(S)$ the (inverse) subsemigroup of regular elements of S. Then

$$\operatorname{Top} \mathbb{K}S \cong \mathbb{K}R(S)$$

Maximal semisimple image (If time allows)

Definition

Let S be an E-Ehresmann semigroup. Denote

$$\mathrm{Reg}_E(S) = \{a \in S \mid a\mathcal{R}a^+, a\mathcal{L}a^*\}$$

This is the set of elements whose corresponding morphism is an isomorphism.

This set was also considered by Lawson; Gould, Zenab.

Remark

If $E = E(S)$ then $\mathrm{Reg}_E(S) = R(S)$.

Maximal semisimple image (If time allows)

Proposition

Let $\mathbb{K}$ be a field with $\operatorname{char}\mathbb{K} = 0$. Let S be a (finite) E-Ehresmann semigroup such that the corresponding Ehresmann category is an EI-category. If $\operatorname{Reg}_E(S)$ is an (inverse) subsemigroup then

$$\operatorname{Top}\mathbb{K}S \cong \mathbb{K}\operatorname{Reg}_E(S)$$

Maximal semisimple image (If time allows)

Lemma

$\text{Reg}_E(S)$ *is a subsemigroup if* S *is left or right restriction (= satisfy the identity* $xy^+ = (xy^+)^+x$ *or* $x^*y = y(x^*y)^*$*)*

Question: Is $\text{Reg}_E(S)$ a subsemigroup for every E-Ehresmann semigroup S? Finite S?

Thank you!