Uncertainty and Synchronization

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Deterministic Finite Automaton

- *DFA* is a triple $A = (Q, \Sigma, \delta)$:
  - Q ... finite set of *states*
  - Σ ... finite set of *letters* (the *alphabet*)
  - δ ... total function $Q \times \Sigma \to Q$ (*transition function*)
- Extended transition function:

$$\delta : 2^Q \times \Sigma^\star \to 2^Q$$

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$$|\delta(Q, w)| = 1$$

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Bounds for Shortest Reset Words

$n = |Q|$

	Full Synchronization
Upper Bounds	$\mathcal{O}\left(n^3\right)$
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Greedy strategies do not work!

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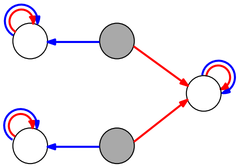

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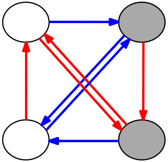

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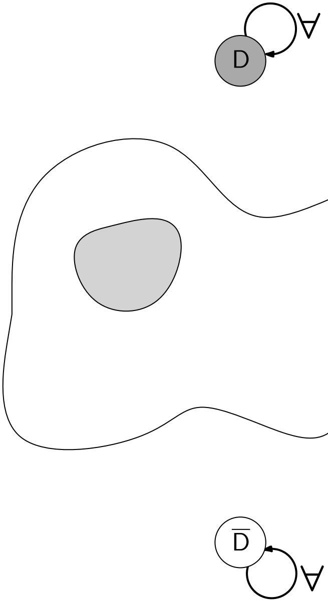
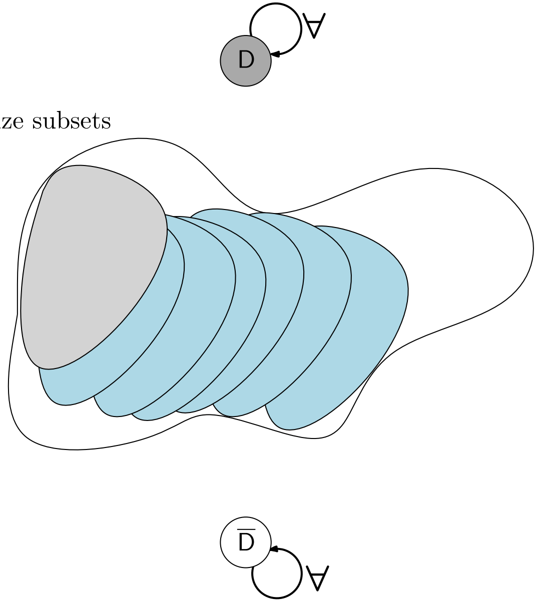


Proofs of the Lower Bound $2^{\Omega(n)}$

letters ≡ half-size subsets



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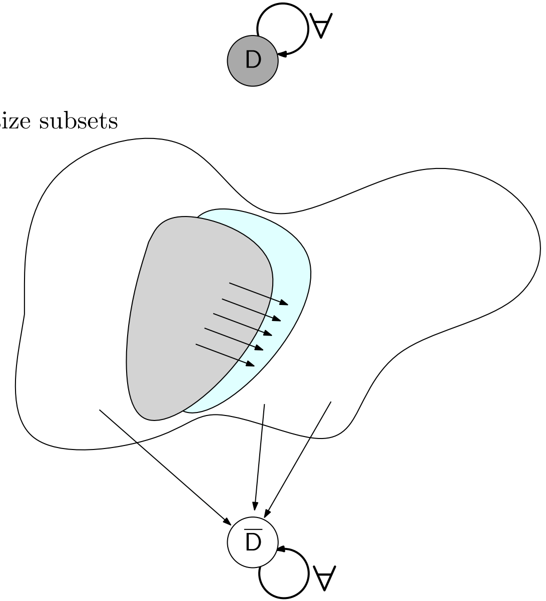

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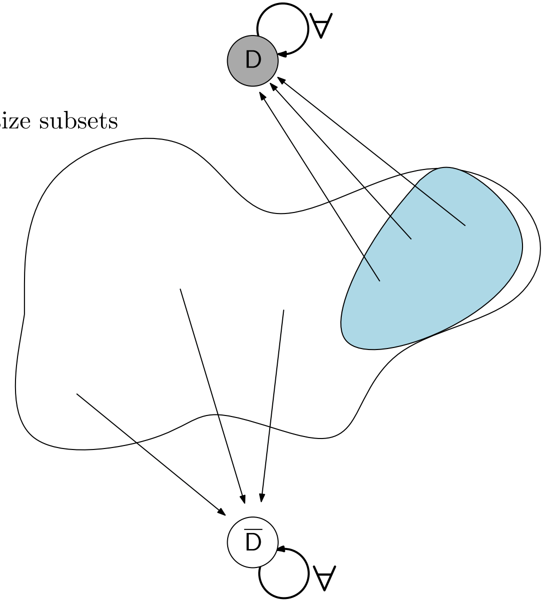

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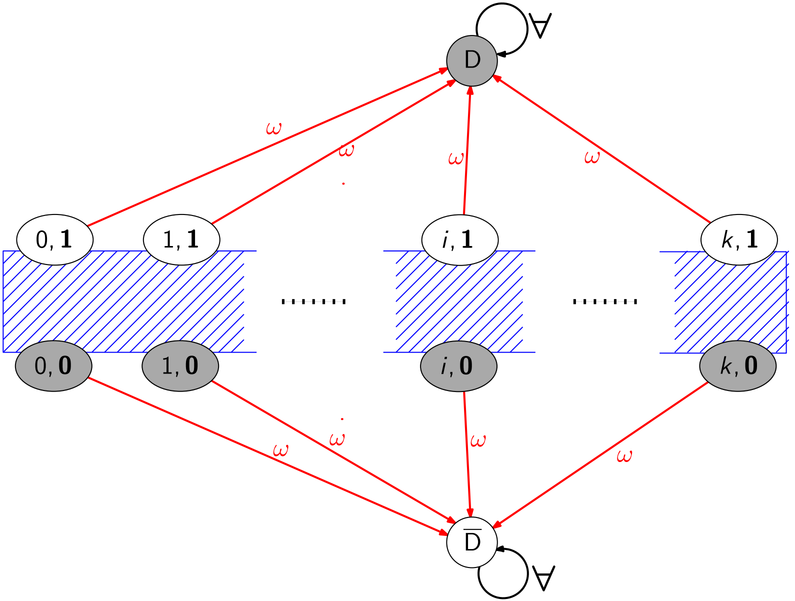

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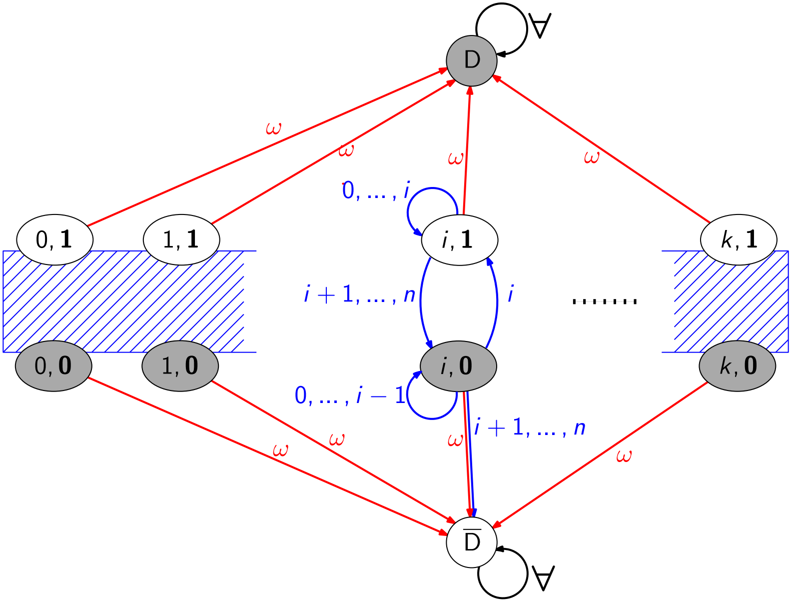

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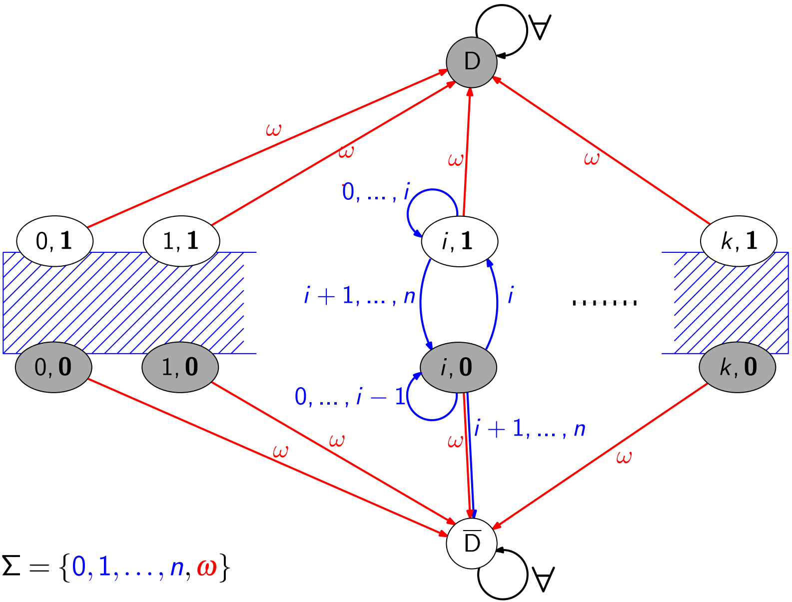
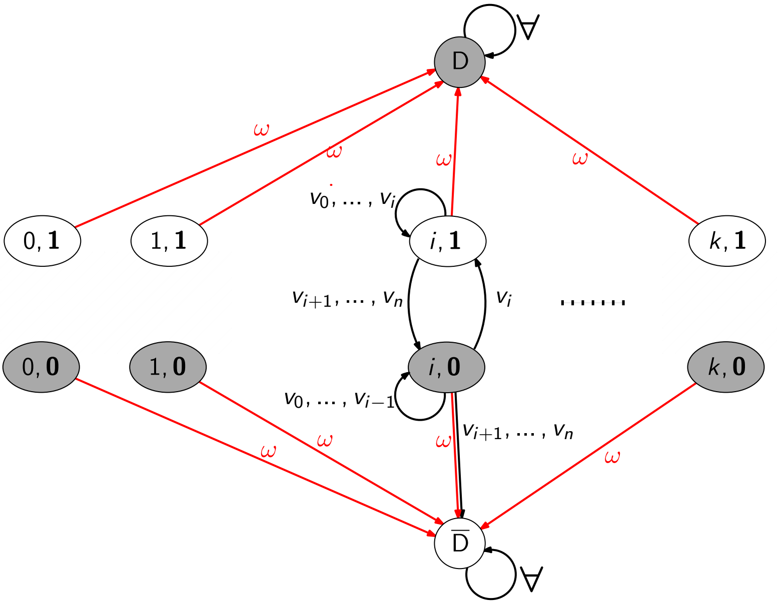


$$\Sigma = \{0, 1, \ldots, n, \omega\}$$



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ω
$v_0, \ldots, v_i$
$0, \mathbf{1}$
$1, \mathbf{1}$
$i, \mathbf{1}$
$k, \mathbf{1}$
$v_{i+1}, \ldots, v_n$
v_i
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$\overline{D}$

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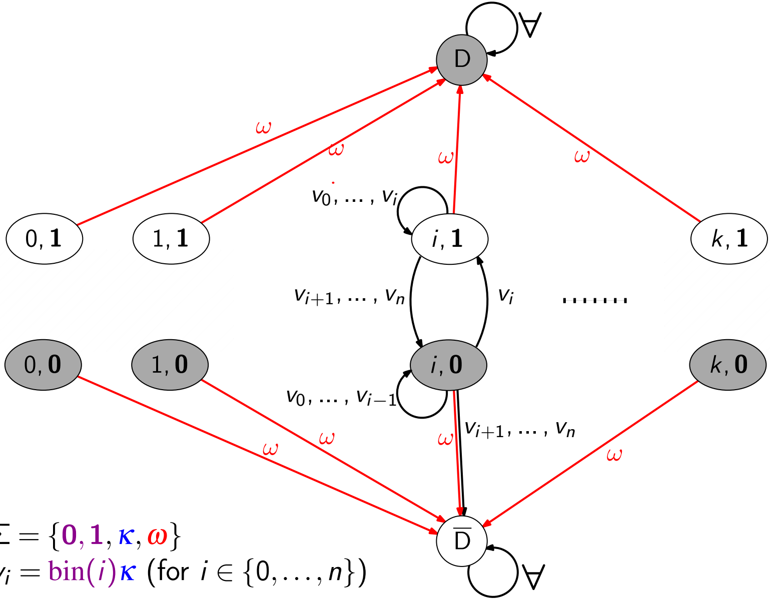

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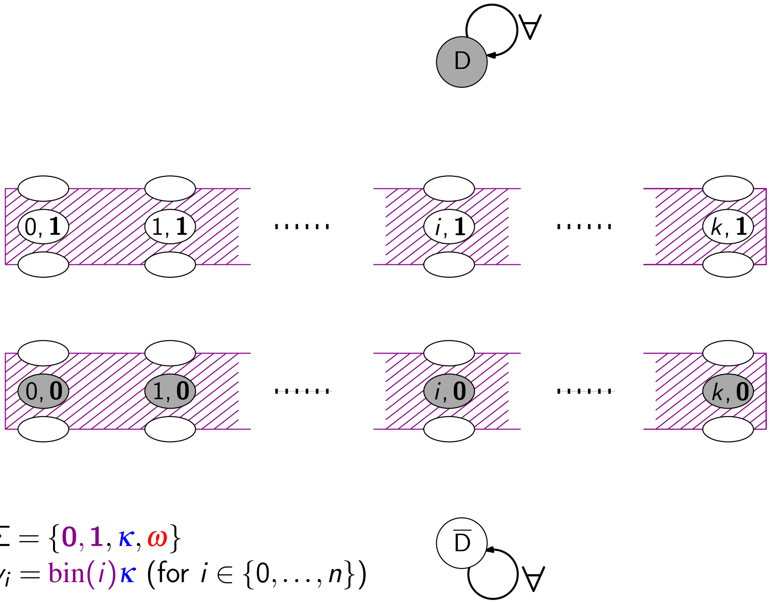

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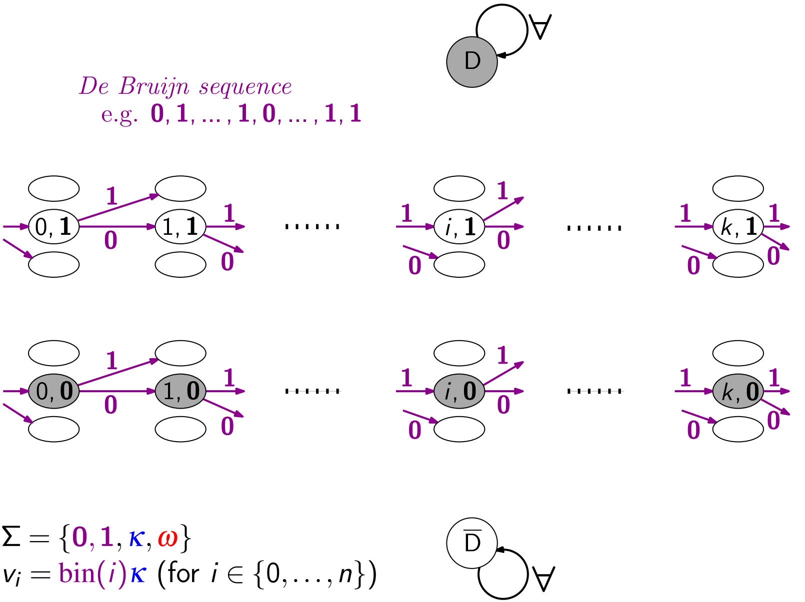

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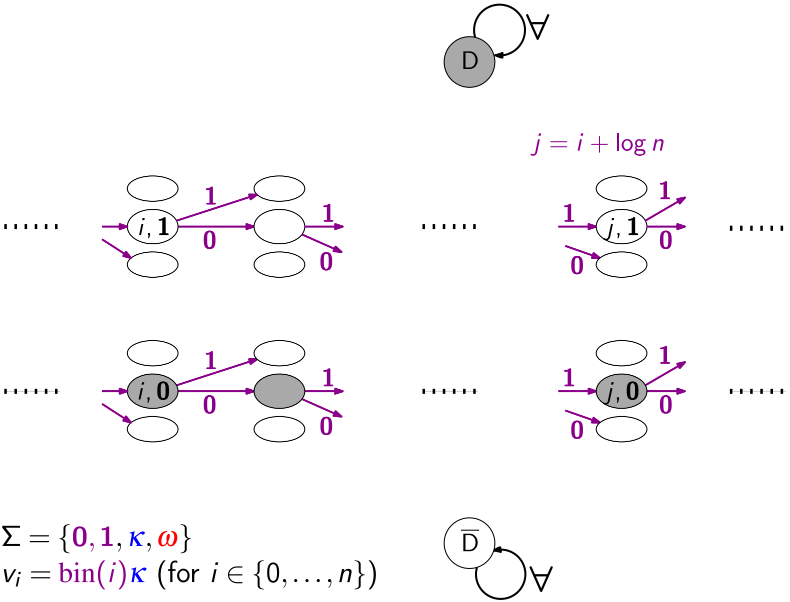

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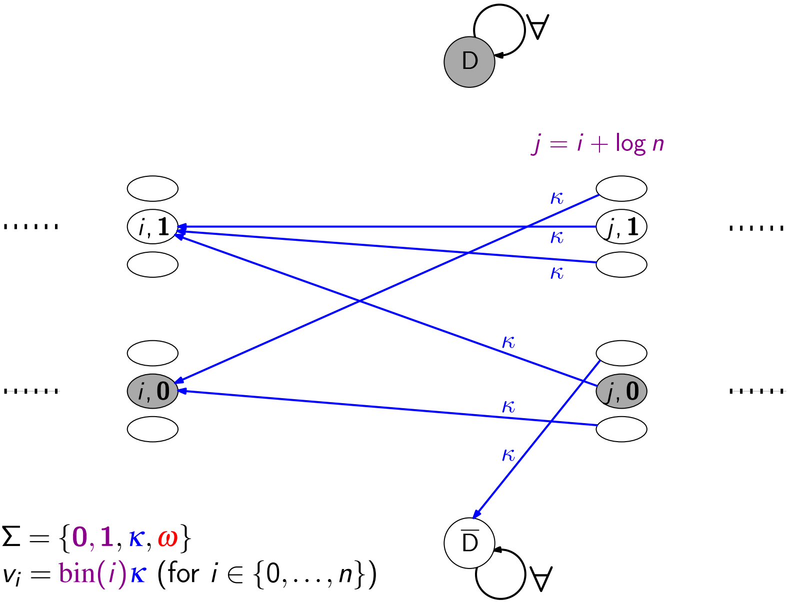

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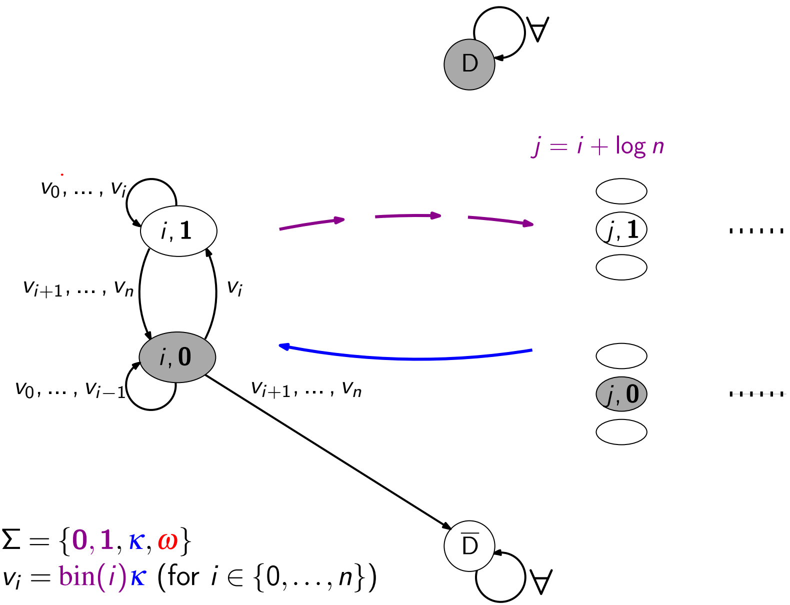

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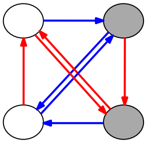
New radix construction	2	no	$2^{\theta(n)}$

Strong Connectivity

A general technique using ***swap congruences***.



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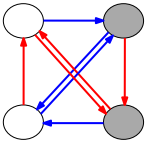

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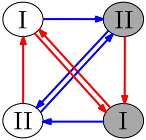

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New radix construction	2	no	$2^{\theta(n)}$
New radix constr. + swapping	2	yes	$2^{\theta(n)}$

Avoiding Words

- $w \in \Sigma^\star$ *avoids* $q \in Q$ if

$$q \notin \delta(Q, w)$$

Bounds for Shortest Avoiding Words

	Synchronizing DFA	General DFA
Upper Bounds	$\mathcal{O}(n^3)$	$2^{\mathcal{O}(n)}$
Lower Bounds	$2n-4$	$2n-4$

Avoiding -> Short Full Reset Words

- Known upper bounds on shortest full reset words:
  - $\frac{1}{3}n^3 + \mathcal{O}(n^2)$ (Kohavi, 1970)
  - $\frac{1}{6}n^3 + \mathcal{O}(n^2)$ (Pin, 1983)
  - $\frac{7}{48}n^3 + \mathcal{O}(n^2)$ if $\mathcal{O}(n)$ for avoiding

Thank you for your attention!